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Economic lot sizing: The capacity reservation model

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ABSTRACT

Capacity reservation contracts allow a consumer to purchase up to a certain capacity at a unit price lower than that of the spot market, while the consumer's excess orders are realized at the spot price. In this paper, we consider a lot sizing problem where the consumer places orders following a capacity reservation contract. In particular, we study the general problem and the polynomial time solvable special cases of the problem and propose corresponding algorithms for them.

1. Introduction

Consider a scenario where a retailer (she) keeps a long-term contract with a manufacturer (he). In return for the retailer's regular purchases, the manufacturer offers the retailer products up to a given capacity at a unit price lower than the spot price. When the retailer's desired procurement quantity exceeds the reservation capacity, she has to realize the excess quantity at the spot price from the same supplier. Known as *capacity reservation contracts*, such contracts are extensively used for purchasing chemicals, commodity metals, semiconductors and electric power [18]. The capacity reservation model can also be applied to production models where extra cost is incurred due to overtime production or the outsourcing part of production.

Formally, we consider the capacity reservation contracts with two parameters, c, the unit purchasing price specified by the capacity reservation contract, and Q, the given capacity. Suppose that the fixed ordering cost is K, the spot price is s, and the retailer's procurement quantity is q. Then, to realize all her demand, the retailer's purchasing cost is

$$Y(q) = \begin{cases} 0 & q = 0 \\ K + cq & 0 < q \le Q \\ K + cQ + s(q - Q) & Q < q \end{cases} \tag{1}$$

or we could equivalently write $Y(q) = K\mathbf{1}_{\{q>0\}} + cq + (s - c)(q - Q)^+$, where $(\cdot)^+ = \max\{\cdot, 0\}$.

Capacity reservation contracts are frequently used in procurement and transportation. For example, Jin and Wu [17] and Erkoc and Wu [10] consider the capacity reservation contracts for the high-tech industry. van Nordena and van de Veldeb [22] first study the dynamic lot sizing problem with a transportation capacity reservation contract. Inderfurth and Kelle [16] consider the combined use of capacity reservation contracts and the spot market in the newsvendor model.

Similar contracts are also studied by several researchers. Henig et al. [14] study a periodic-review inventory-control model. In their model, when the order quantity is below a given volume, the ordering cost is zero, otherwise the cost is linear in the exceeding quantity. Chao and Zipkin [4] consider a similar problem, where a fixed cost is incurred if the order quantity is above the volume. Caliskan-Demirag et al. [3] study a periodic-review inventory problem where the fixed cost depends on the order quantity.

Our work, on the other hand, is an extension of the economic lot sizing problem, with deterministic and time varying demand, capacity and cost parameters. The goal is to find a plan that minimizes the total inventory and procurement cost. Wagner and Whitin [24] first develop an $O(T^2)$ dynamic programming algorithm for the general lot sizing problem, also known as the WW problem. Later research works focus on studying algorithm complexity for different models; see [20] for example. The $O(T^2)$ dynamic programming algorithm was improved independently by Aggarwal and Park [1], Federgruen and Tzur [11] and Wagelmans et al. [23] who developed an $O(T \log T)$ algorithm for the general problem. The capacitated lot sizing problem (CLSP) can be viewed as a generalization of the WW problem. Known to be $\mathcal{NP}$-hard [2,13], many heuristics are designed for the CLSP problem [8,15]. Readers might refer to [9] for a survey of lot sizing problems.

The capacity reservation model discussed in this paper could be conveniently viewed as an extension of the CLSP problem, since an algorithm for the lot sizing problem with capacity reservation (LS-CR) could always be applied to a capacitated lot sizing problem by setting the spot price to infinity at every time slot. The LS-CR

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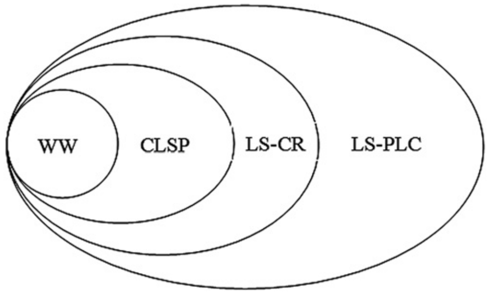

Fig. 1. The relationship between different problems.

Table 1
Complexity of different models.

	$NI/G/NI/ND$	$G/G/G/C$
Traditional model	$O(T^2)$ [a]	$O(T^3)$ [b]
Our model	$O(T^3)$	$O(T^4)$

[a] See [6] for details.
[b] See [22] for details.

problem, on the other hand, is a special case of the lot sizing problem with piecewise linear production costs (LS-PLC). Chen et al. [5] present an efficient dynamic programming algorithm for the general LS-PLC problem with computational results, and Shaw and Wagelmans [19] propose a pseudo-polynomial time algorithm. We characterize the relationship between the different lot sizing problems in Fig. 1.

Although the capacitated problems are quite difficult to solve in general, many special cases have polynomial time algorithms. Bitran and Yanasse [2] design a classification scheme for the capacitated lot sizing problem, and introduce the four field notation $\alpha/\beta/\gamma/\delta$, where α, β, γ and δ represent the setup cost (fixed ordering cost in our model), unit holding cost, unit production cost (unit purchasing cost from long-term contract and spot market price in our model), and capacity type (reservation quantity in our model), respectively. Each of the parameters might have an arbitrary pattern (general, G), be constant (C), nondecreasing (ND), nonincreasing (NI) or zero (Z). Different from the scheme, we allow unlimited purchase from the spot market (at a potentially higher price) when the reservation quantity is exhausted, and thus a feasible solution always exists. It should be noted that when $\gamma = C$ (resp., NI, ND), both the unit purchasing prices specified in the capacity reservation contract and the spot prices are stationary (resp., nonincreasing, nondecreasing) over all time periods.

In this paper, we focus on the following two models: $NI/G/NI/ND$ and $G/G/G/C$, both of which are known to have polynomial time algorithms in the classic model but not known in the capacity reservation model. In this work, we show that both models have polynomial time algorithms in the capacity reservation model, and we present our results in Table 1.

2. Problem formulation and computational complexity

Consider the problem with T horizons, and the capacity reservation contract specifies unit purchasing cost c_i and capacity q_i for period $i \in \{1, \ldots, T\}$. The inventory holding cost for period i is h_i, while backorders are not allowed. The demand in period i is d_i, which is known prior to the making of the purchase decision. In the general model, the fixed ordering cost for period i is K_i. The spot price in time i is s_i, where $s_i > c_i$, implying that the spot price is always higher than price within the given capacity. All costs are nonnegative and the capacity is strictly positive.

The decision variables q_i are the purchasing quantities in periods i, $i = 1, \ldots, T$, and the objective is to minimize the purchasing costs and inventory costs,

$$\min \sum_{i=1}^{T} K_i 1_{\{q_i>0\}} + c_i q_i + (s_i - c_i)(q_i - Q_i)^+ + h_i I_i$$

where I_i represents the inventory level at the end of period i. The program is subject to the following constraints:

- Balance of the inventory:

  $I_{i-1} + q_i - d_i = I_i.$
- Nonnegativity:

  $q_i \geq 0, \qquad I_i \geq 0.$
- And the initial inventory level is zero:

  $I_0 = 0.$

The dynamic lot sizing problem without fixed ordering cost is the case $K_i = 0$ for every i.

The general lot sizing problem with capacity reservation is $\mathcal{NP}$-hard due to the $\mathcal{NP}$-hardness of its special case, i.e., the capacitated lot sizing problem. In fact, the problem remains $\mathcal{NP}$-hard even if we put stronger restrictions on the problem, including the classes $C/Z/NI/NI$, $C/Z/ND/ND$, $ND/Z/Z/ND$, $NI/Z/Z/NI$, $C/G/Z/NI$ and $C/C/ND/NI$; see [2] for proofs.

A straightforward dynamic programming algorithm could solve the general problem in time $O(T^3\overline{d}^2)$, where $\overline{d}$ denotes the average demand over all periods.

More efficient algorithms could be explored for the general problem. Shaw and Wagelmans [19] show that the capacitated lot sizing problem with piecewise linear production cost functions can be solved in $O(T^2\overline{q}\overline{d})$ time, where $\overline{q}$ is the average number of pieces needed to represent the linear cost function.

As discussed before, it would be convenient to view the lot sizing problem in the capacity reservation model as a special case of the piecewise cost function model; therefore we immediately come up with the following result: there exists an $O(T^2\overline{d})$ algorithm for the lot sizing problem in the capacity reservation model. Since the general problem is $\mathcal{NP}$-hard, this pseudo-polynomial time algorithm is quite useful.

When there is no fixed ordering cost, the lot sizing problem is solvable in time $O(T \log T)$ by simply keeping a sorted list of available purchasing alternatives.

3. The $NI/G/NI/ND$ class reservation problem

Although the lot sizing problem in the capacity reservation model is $\mathcal{NP}$-hard in general, it is still possible to solve some special cases of the problem quite efficiently. Bitran and Yanasse [2] first design an $O(T^4)$ algorithm for the $NI/G/NI/ND$ capacitated lot sizing problem, and Chung and Lin [6] improve their result to $O(T^2)$. Motivated by their work, we study the $NI/G/NI/ND$ class problem in the capacity reservation model.

For the $NI/G/NI/ND$ problem, over the time, the setup costs K_i are nonincreasing, the unit holding costs have arbitrary pattern, the unit purchasing cost c_i and s_i are nonincreasing and the capacities Q_i are nondecreasing. The traditional approaches (cf. [6]) for the capacitated lot sizing problem use the idea of a subplan, and so do we.

Definition 1. A subplan S_{uv} is the part of plan covering the demand from $u + 1$ to v such that $I_u = 0, I_v = 0$ and $I_t > 0$ for any $u < t < v$.

To develop our algorithm for the $NI/G/NI/ND$ class, observe that

$$f(t) = \min_{1 \leq u < t} \{f(u) + C(S_{ut})\} \tag{2}$$

where $C(S_{ut})$ is used to denote the minimum cost of the subplan S_{ut}, while $f(t)$ represents the optimal objective function value for the subplan including the first t periods, with $I_0 = I_t = 0$. In the optimal solution, the remaining inventory at the end of the planning horizon should be zero, i.e., $I_T = 0$, so our objective is to solve the T-period problem $f(T)$. It follows straightforwardly that if we could compute the optimal solution for every subplan S_{uv} efficiently, the whole problem would be solved.

There may be multiple optimal plans that minimize the total cost. To break ties, we use a tie-breaking rule called minimality of sum. That is, if there are multiple optimal solutions, we pick the one that minimizes the sum $\sum_{i=1}^{T}(q_i + I_i)$. The same tie-breaking rule is used in [7].

Theorem 1. *For any optimal plan that minimizes the sum $\sum_{i=1}^{T}(q_i + I_i)$, any of its subplans S_{uv}, with $u + 1 = i_1 < \cdots < i_k \le v$ being ordering points to cover the demand from $u + 1$ to v, would tolerate only one of the following cases.*

- *Case* 1. $q_{i_1} \le Q_{i_1}$ *and* $q_{i_t} = Q_{i_t}$ *for every* $1 < t \le k$, *or*
- *Case* 2. $q_{i_k} \ge Q_{i_k}$ *and* $q_{i_t} = Q_{i_t}$ *for every* $1 \le t < k$.

Proof. To prove the theorem, it suffices to show the following: (1) if $q_{i_t} < Q_{i_t}$, then $t = 1$; (2) if $q_{i_t} > Q_{i_t}$, then $t = k$; (3) $q_{i_1} < Q_{i_1}$ and $q_{i_k} > Q_{i_k}$ cannot happen simultaneously.

Let us start with (1). Assume that $q_{i_t} < Q_{i_t}$ for some $t > 1$. If there are more than one such points, let t be the smallest value such that $1 < t$ and $q_{i_t} < Q_{i_t}$. We consider a new plan for the whole problem by changing $q_{i_1} \leftarrow q_{i_1} - 1$ and $q_{i_t} \leftarrow q_{i_t} + 1$; denote the new plan by q'. Since $I_i > 0$ for any $i_1 \le i < i_t$ in the original plan, the new plan is feasible. In addition, based on the definition of $NI/G/NI/ND$, we have $s_{i_1} > c_{i_1} \ge c_{i_t}$, which implies that the new plan costs no more than the old plan. If the new plan costs less, then the optimality of the original plan is violated. Otherwise, because our perturbation deceases q_{i_1} by 1, increases q_{i_t} by 1, and decreases I_x by 1 for any $i_1 \le x < i_t$, the original plan violates the supposed minimality of sum. This completes the proof.

As for (2), assume that $q_{i_t} > Q_{i_t}$ for some $t < k$ (if there are multiple such points, let t be the largest one), and consider a new plan for the whole problem by changing $q_{i_t} \leftarrow q_{i_t} - 1$ and $q_{i_k} \leftarrow q_{i_k} + 1$. Since $I_i > 0$ for any $i_t \le i < i_k$ in the original plan, the new plan is feasible. Note that $s_{i_t} \ge s_{i_k} > c_{i_k}$; the new plan is at least as good as the original one. Similar to (1), if the new plan costs less, then the optimality of the original plan is violated. If both plans have the same cost, the new plan has smaller sum $\sum_{i=1}^{T}(q_i + I_i)$; hence the original plan violates the supposed minimality of sum.

As for (3), we decrease q_{i_1} by 1 and increase q_{i_k} by 1, and denote the new plan q' (the feasibility of the new plan follows from the property of subplan); because of the optimality of q, we have the following result:

$$c_{i_1} + \sum_{j=i_1}^{i_k - 1} h_j \le s_{i_k}.$$

Similarly, we increase q_{i_1} by 1 and decrease q_{i_k} by 1, and denote the new plan q''. Still, the optimality of q yields the following result immediately:

$$c_{i_1} + \sum_{j=i_1}^{i_k - 1} h_j \ge s_{i_k}.$$

Therefore we must have

$$s_{i_k} \le s_{i_1} + \sum_{j=i_1}^{i_k - 1} h_j \le s_{i_k}.$$

So the two inequalities must be tight, i.e., $c_{i_1} + \sum_{j=i_1}^{i_k-1} h_j = s_{i_k}$. Therefore, both q' and q'' are optimal plans. However, as q' has smaller sum $\sum_{i=1}^{T}(q_i + I_i)$, the supposed minimality of sum is violated in the original plan q. □

Based on Theorem 1, we could compute the optimal strategy for each subplan by considering the two cases respectively, and pick the cheaper one as the optimal solution. To solve each case, the property of the class requires us to order as late as possible, as the following lemma shows.

Lemma 1. *There exists an optimal production plan such that for any subplan S_{uv}, if j is an ordering point and $j + 1$ is not an ordering point, then deferring the orders placed at j to $j + 1$ would make the plan infeasible.*

Proof. Otherwise, if the new plan is still feasible, since $j+1$ has less fixed ordering cost, less procurement cost and larger capacity, the new plan is at least as good as the previous one. We could repeat this process until all orders could not be deferred, and the plan is our desired one. □

The optimal strategy corresponding to Case 1 could be computed using the algorithm presented by [6], and to be complete, we briefly introduce their approach here.

Beginning with $i_{k+1} = v + 1$, inductively pick $i_t < i_{t+1}$. Following Lemma 1, the property of i_t is that it is the largest $i < i_{t+1}$ such that the capacity $Q_{i_k} + \cdots + Q_{i_{t+1}} + Q_i$ cannot cover the demand from period i to v. The optimal Case 1 strategy could be computed in time $O(T^2)$ for all subplans.

Consider Case 2 where we order in each subplan exactly the capacity at every ordering point except the last one, at which we could order more than the capacity. Then, given the ordering points $i_1, \dots, i_k$, the ordering strategy can be uniquely determined, so all we need to do is to find the ordering points.

To find the ordering points, we enumerate the last ordering point in the subplan, i.e., i_k. Given a subplan S_{uv}, we first compute a standard strategy.

Algorithm 1.
- *Step* 1. Start with $i = u$ and $q_j = 0$ for every $u + 1 \le j \le v$.
- *Step* 2. Find the smallest j with $i < j \le v$ such that $I_j < 0$; if no such j exists, the algorithm completes.
- *Step* 3. Set $i = j$. For $k = j, j-1, \dots, u+1$: set $q_k = Q_k$; if $I_j \ge 0$, immediately goto Step 2, otherwise continue the process until $I_j \ge 0$.

Let $j_1 < \cdots < j_m$ be the ordering points obtained using Algorithm 1. To better illustrate the idea, consider the following subplan S_{05}. The demand for day 1 to 5 is respectively given by $d_1 = 10$, $d_2 = 10$, $d_3 = 40$, $d_4 = 10$ and $d_5 = 10$. The capacity is the same for all days with $Q_i = 25$, and initialize the order to $q_i = 0$. Since $I_1 = -10 < 0$, we update the subplan to $(q_1, q_2, q_3, q_4, q_5) = (25, 0, 0, 0, 0)$. Now, we have $I_1 = 15$, $I_2 = 5$ and $I_3 = -35 < 0$. Then, we update the solution to $(q_1, q_2, q_3, q_4, q_5) = (25, 0, 25, 0, 0)$. Note that $I_3 = -10$ is still below zero. Then, we update the solution to $(q_1, q_2, q_3, q_4, q_5) = (25, 25, 25, 0, 0)$, and $I_3 = 15$, $I_4 = 5$ and $I_5 = -5 < 0$. Then, by setting $(q_1, q_2, q_3, q_4, q_5) = (25, 25, 25, 0, 25)$ we have $I_5 > 0$. Hence, the output of the algorithm is $j_1 = 1, j_2 = 2, j_3 = 3$ and $j_4 = 5$.

Now we claim the following.

Theorem 2. *For Case 2 described above, there exists an optimal solution q of subplan S_{uv} such that its ordering points can be denoted as $j_1, \dots, j_k$ for some $k \le m$.*

Proof. First we note that Theorem 2 is equivalent to the statement that "for some optimal subplan with ordering point $i_1 < \cdots < i_k$, there is no $1 \le t \le k$ such that $i_t \ne j_t$".

Now consider an optimal subplan where $1 \le t \le k$ is the first point such that the statement is violated, i.e., $i_t \ne j_t$. Note that $i_t < j_t$, otherwise the subplan would be infeasible.

To show this claim, assume that j_v is the first among j's such that j_v is the first non-ordering point in the optimal solution, i.e., $j_v \notin \{i_1, \dots, i_k\}$ and $\{j_1, \dots, j_{v-1}\} \subset \{i_1, \dots, i_k\}$. If there is no such j_v, let $j_v = j_m$. Consider an alternative solution q' obtained by setting $q'_{i_t} \leftarrow 0$ and $q'_{j_v} \leftarrow q_{i_t}$, $q'_i \leftarrow q_i$ for other points. That is, q' is obtained by moving the order from i_t to j_v in q, $i_t < j_t \leq j_v$. Based on the property of *NI/G/NI/ND*, we have $C(q') \leq C(q)$, and it suffices to show that q' is feasible. Since $j_1, \ldots, j_v$ is feasible for points up to j_v, and $j_1, \ldots, j_v$ are all ordering points in q', so q' is feasible for points up to j_v.

If in q', there is still some $i_{t'} \neq j_{t'}$, we repeat the above process until the theorem is satisfied. This proves the theorem. □

For any subplan, it suffices to study the following candidates for the optimal solution: $j_1 < \cdots < j_k, k = 1, \ldots, m$, where $m \leq T$. When the cost of solution $j_1, \ldots, j_t$ is known, the cost of $j_1, \ldots, j_{t+1}$ could be computed in time $O(1)$. As there are $O(T^2)$ subplans, it takes $O(T^3)$ to find the Case 2 solutions to all subplans. Based on (2), we have the following theorem.

Theorem 3. *There is an $O(T^3)$ algorithm for the class NI/G/NI/ND.*

4. The *G/G/G/C* class reservation problem

In this section, we consider the economic lot sizing problem with constant capacities. The original problem without capacity reservation was known to be solvable in $O(T^4)$ [12], and van Hoesel and Wagelmans [21] later improved the algorithm to solve the problem in time $O(T^3)$.

Since the capacity constraints (which are reservation quantities here) are constant over all periods, in this section, we use Q to replace Q_i for every time slot i.

Theorem 4. *For any optimal plan that minimizes the sum $\sum_{i=u+1}^{T} (q_i + I_i)$, any of its subplans S_{uv}, with $u + 1 = i_1 < \cdots < i_k \leq v$ being ordering points to cover the demand from $u+1$ to v, its optimal solution $(q_{u+1}, \ldots, q_v)$ should have the following property: if $q_{i_x} \neq Q$ for some $1 \leq x \leq k$, then $q_{i_y} = Q$ for every $1 \leq y \leq k, y \neq x$.*

Proof. Suppose that there exists an optimal solution to the subplan S_{uv} such that there exist two different ordering points $1 \leq x < y \leq k$ such that $q_{i_x} \neq Q$ and $q_{i_y} \neq Q$, and among all optimal plans, it minimizes $\sum_{i=u+1}^{T}(q_i + I_i)$. As q_{i_x} and q_{i_y} are both ordering points, we have $q_{i_x} > 0$ and $q_{i_y} > 0$.

We consider a new plan q' by changing $q_{i_x} \leftarrow q_{i_x} - 1$ and $q_{i_y} \leftarrow q_{i_y} + 1$, and another new plan q'' by changing $q_{i_x} \leftarrow q_{i_x} + 1$ and $q_{i_y} \leftarrow q_{i_y} - 1$. Using the property of subplan, we figure out immediately that both new plans are feasible solutions. Since $C(q) \leq C(q')$, $C(q) \leq C(q'')$, we have $c_{i_x} + \sum_{j=i_x}^{i_y-1} h_j \leq s_{i_y}$ and $c_{i_x} + \sum_{j=i_x}^{i_y-1} h_j \geq s_{i_y}$; therefore both inequalities are tight and we have $C(q) = C(q') = C(q'')$, so in terms of costs, q' is as good as q, i.e., they are both optimal plans. However, by definition, q' has smaller sum $\sum_{i=u+1}^{T}(q_i + I_i)$; hence the supposed minimality of sum property is violated in q.

Based on Theorem 4, we immediately have the following corollary.

Corollary 1. *For the G/G/G/C problem in the capacity reservation model, there exists an optimal solution such that for every subplan S_{uv}, the ordering points $u + 1 = i_1 < \cdots i_k \leq v$ satisfy that*

- *There exists a $1 \leq t \leq k$ such that $q_{i_t} = d_{uv} - (k-1)Q$, where $d_{u,v}$ denotes the demand from u to $v - 1$.*
- *For every $1 \leq x \leq k$ and $x \neq t$, we have $q_{i_x} = Q$.*

Further, we could also show the following.

Corollary 2. *For the G/G/G/C problem in the capacity reservation model, there exists an optimal solution such that for every subplan S_{uv} and any $u + 1 \leq k \leq v$ we have that*

- $\sum_{i=u+1}^{k} q_i \bmod Q = 0$ *or* $\sum_{i=u+1}^{k} q_i \bmod Q = \Delta$,
- $\lfloor \sum_{i=u+1}^{k} q_i/Q \rfloor \leq v - u$ *or* $\lfloor \sum_{i=u+1}^{k} q_i/Q \rfloor \geq \lfloor d_{u,v}/Q \rfloor - v + u$,

where $\Delta = d_{u,v} \bmod Q$.

Note that if there exists an $i \leq k$ such that $q_i \bmod Q = \Delta$, then $(\sum_{k+1}^{v} q_i)/Q \leq (v-k) \leq (v-u)$ and hence $\lfloor \sum_{i=u+1}^{k} q_i/Q \rfloor \geq \lfloor d_{u,v}/Q \rfloor - v + u$. Otherwise, $\sum_{u+1}^{k} q_i \leq (k-u) \leq (v-u)$ and hence $\lfloor \sum_{i=u+1}^{k} q_i/Q \rfloor \leq v - u$.

Based on Corollary 2, for any subplan S_{uv}, we call $u+1 \leq k \leq v$ a type 0 point with order j iff $\sum_{i=u+1}^{k} q_i \bmod Q = 0$ and $\sum_{i=u+1}^{k} q_i/Q = j$; call it a type 1 point with order j iff $\sum_{i=u+1}^{k} q_i \bmod Q = \Delta$ and $\lfloor \sum_{i=u+1}^{k} q_i/Q \rfloor = j$. Let $C(k, x, j)$ denote the minimum cost from $u+1$ to k, given that k is a type x point with order j. $C(k, x, j)$ is computed using dynamic programming in the following way. Note that in the following process we are only interested in j such that $j \leq v - u$ or $\lfloor d_{u,v}/Q \rfloor - v + u \leq j \leq \lfloor d_{u,v}/Q \rfloor$, as shown in Corollary 2. $j \leq \lfloor d_{u,v}/Q \rfloor$ holds intuitively because we would not order more than the actual demand.

- For a type 0 point, we have
$$C(k,0,j) = \min\{C(k-1,0,j) + (jQ - d_{u,k-1})h_{k-1}, C(k-1,0,j-1) + (jQ - d_{u,k-1} - Q)h_{k-1} + K_k + Qc_k\}.$$
- For a type 1 point, we have four subtypes, and the cost is minimized over the subtypes, $C(k,1,j) = \min_{1 \leq i \leq 4} C_i(k,1,j)$.
  - Subtype 1.1: $q_k = Q$ or $q_k = 0$, which implies that in the solution, we order nothing or Q in time slot k, and hence the cost $C_1(k,1,j)$ is the minimum over the two cases:
  $$C_1(k,1,j) = \min\{C(k-1,1,j) + (jQ + \Delta - d_{u,k-1}) \times h_{k-1}, C(k-1,1,j-1) + (jQ + \Delta - d_{u,k-1} - Q)h_{k-1} + K_k + Qc_k\}.$$
  - Subtype 1.2: $q_k = \Delta$, which implies that in the solution, we order Δ in time slot k:
  $$C_2(k,1,j) = C(k-1,0,j) + (jQ - d_{u,k-1})h_{k-1} + K_k + c_k\Delta.$$
  - Subtype 1.3: $q_k = Q + \Delta$, which implies that in the solution, we order $Q + \Delta$ in time slot k:
  $$C_3(k,1,j) = C(k-1,0,j-1) + (jQ - d_{u,k-1} - Q) \times h_{k-1} + K_k + s_k\Delta + Qc_k.$$
  - Subtype 1.4: $q_k > Q + \Delta$, which implies that in the solution, we order $tQ + \Delta$ in time slot k for some $t > 1$. To make the computation more efficient, we use a dynamic programming to cumulate the cost from $(t-1)Q + \Delta$ to $tQ + \Delta$, as follows:
  $$C_4(k,1,j) = \min\{C_3(k,1,j-1) + Qs_k, C_4(k,1,j-1) + Qs_k\}.$$
- For any k, we still need to check the feasibility of the plans. If $jQ + \Delta < d_{u,k}$, then the plan is infeasible and $C(k,i,j)$ should be reset to $+\infty$.

Theorem 5. *There is an $O(T^4)$ algorithm for the class $G/G/G/C$.*

Proof. First, for each subplan, our goal is to compute $C(v, 1, \lfloor d_{u,v}/Q \rfloor)$, the minimum cost. The simple dynamic programming algorithm is solvable in time $O(T^2)$; hence every subplan could be optimally computed in $O(T^2)$. There are $O(T^2)$ subplans, and so it takes $O(T^4)$ to solve all the subplans. Then we could compute the optimal solution to the global problem using the shortest path algorithm, which takes an additional time $O(T^2)$. Therefore the global problem can be solved in $O(T^4)$. □

5. Conclusion and future research

The capacity reservation model is a generalization of the capacitated economic lot sizing problem. Despite the $\mathcal{NP}$-hardness of solving the general model, in this paper, we are able to solve two special cases, *NI/G/NI/ND* and *G/G/G/C* with time $O(T^3)$ and $O(T^4)$, respectively.

There are several interesting problems to be explored in the future. Although it is straightforward to see that the capacity reservation model is harder than the capacitated lot sizing problem, we

Table 2
Computational complexity of other polynomial time solvable classes.

	$NZ/D/NZ/NI$	$C/Z/C/G$
Capacitated lot sizing	$O(T)$	$O(T \log T)$
Capacity reservation model	$O(T)$	$O(T^2)$

are not able to separate the two complexity classes, i.e., we are not sure if there exists a class $\alpha/\beta/\gamma/\delta$ which has polynomial time algorithms in the capacitated lot sizing model, but is $\mathcal{NP}$-hard in the capacity reservation model. If the answer is no, could we find a general approach to construct polynomial time algorithms for the capacity reservation model from traditional models? It is not known if there exists a model $\alpha/\beta/\gamma/\delta$ such that the two models have different computational complexities, if they are both polynomial time solvable. Table 2 summarizes the computational complexity of other classes that are known to be solvable in polynomial time given by [2].

Except from those theoretical issues, finding practical algorithms for other classes and improving current algorithms is interesting. It also remains open to generalize the result to the piecewise linear cost function model.

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