The Loss-Averse Newsvendor Problem with Supply Options

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Abstract: In this article, we consider a loss-averse newsvendor with stochastic demand. The newsvendor might procure options when demand is unknown, and decide how many options to execute only after demand is revealed. If the newsvendor reserves too many options, he would incur high reservation costs. Yet reserving too few could result in lost sales. So the newsvendor faces a trade-off between reservation costs and losing sales. When there are multiple options available, the newsvendor has to consider how many units of each to reserve by studying the trade-off between flexibility and costs. We show how the newsvendor's loss aversion behavior affects his ordering decision, and propose an efficient algorithm to compute his optimal solution in the general case with n options. We also present examples showing how the newsvendor's ordering strategy changes as loss aversion rises. © 2014 Wiley Periodicals, Inc. Naval Research Logistics 62: 46–59, 2015

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1. INTRODUCTION

We consider a retailer who, as in the newsvendor problem, faces a random demand volume over a single sales season. To cover the random demand, the retailer may, at the beginning of the season, avail herself of various supply contracts, offered by the same or different suppliers. A supply contract is characterized by two parameters: a per unit reservation price and a per unit execution price. Within each contract, the retailer selects the reservation quantity and pays the reservation price for each reserved unit. After the season's demand is revealed, the contract allows the retailer to request delivery of any desired amount up to the reserved quantity, at the additional per unit execution price. The retailer faces the problem of determining an optimal combination of reservation quantities, one for each of the available contracts.

This problem was studied by Martínez-de-Albéniz and Simchi-Levi [24, 25], but under the standard assumption that the retailer is risk neutral and wishes to maximize her expected profit. Behavioral economics, in particular prospect theory (see [31], e.g.), has demonstrated that many decision makers are loss averse: this means that losses are weighted much more than gains. A standard way to represent the loss aversion phenomenon is to assume that a loss is multiplied by a factor $\lambda \geq 1$, while a positive gain is taken as is. Clearly, the larger λ is, the more loss averse the decision maker. When $\lambda = 1$, we have the risk-neural case. The above represents a concave piecewise linear utility function with two linear pieces. We extend our analysis to more general, concave piecewise linear utility functions. This allows us to cover risk-neutral firms facing piecewise linear taxes. It is worth noting that we are dealing with a more complex problem than that in [25], because we are faced with quasilinear utility curves and *ex ante* we do not know which piece the real solution falls on.

We conduct a numerical study that generates the following intuitive insight: the larger the loss aversion, the more the decision maker is inclined to use option contracts with relatively low reservation prices. Even though the total per unit procurement cost under these contracts is higher, the upfront reservation payment is reduced.

The rest of this article is organized as follows. Section 2 reviews the related literature. In Section 3, we introduce and model the problem. Section 4 studies the problem when there is only one option available. Section 5 generalizes the single-option case to the general case with n options, assuming that the set of active options are known and the cumulative demand function takes the form of a piecewise linear function. A polynomial algorithm is proposed to compute

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the optimal solution. In Section 6, we show how to find the active options. Section 7 further considers the case where the cumulative demand is any increasing function and the case where the utility function is piecewise linear. Section 8 concludes the article.

2. LITERATURE REVIEW

The newsvendor problem has attracted plenty of research interest. We refer readers to [21] for a survey of the related research. Various applications of the newsvendor problem have been found. For example, Prasad et al. [28] consider the newsvendor problem in advance selling, Koulamas [22] studies the newsvendor problem with revenue sharing and channel coordination, and Olivares et al. [26] present a structural estimation framework for the newsvendor problem with an application to reserving operating room time.

Recently, a large number of attempts have been made to extend the standard newsvendor model. A recent survey of the joint inventory and pricing newsvendor models can be found in [7]. Ding et al. [12] and Yue et al. [36] study the newsvendor model in which the demand distribution is only partially known. Granot and Yin [17, 18] analyze the price-dependent newsvendor model. Petruzzi et al. [27] study the newsvendor problem with a consumer search cost. Wang [32] and Wu et al. [35] study the modified newsvendor model with advertising.

Some researchers have attempted to consider the decision bias of the newsvendor, which is defined as the behavior to deviate from the optimal order quantity. Eeckhoudt et al. [13] are the first to show that a risk-averse newsvendor would order strictly less than a risk-neutral newsvendor. Agrawal and Seshadri [1] consider a risk-averse retailer that makes order quantity and selling price decisions with the objective of maximizing expected utility. Su [30] analyzes the newsvendor problem where newsvendors are prone to errors and biases. Chen et al. [9] consider a risk-averse newsvendor model with stochastic price-dependent demand. Choi et al. [10] consider a multiproduct risk-averse newsvendor under the law-invariant coherent measures of risk. Risk-aversion is also studied in multiperiod inventory management [8, 37].

Wang and Webster [34] use loss aversion to consider the newsvendor's behavior in the single-period newsvendor problem, where an emergency delivery with higher cost is made when a shortage occurs. Wang [33] considers a loss-averse newsvendor game where multiple loss-averse newsvendors compete for inventory from a risk-neutral supplier.

There are several papers considering flexible contracts instead of firm ones. Dada et al. [11] study a newsvendor procurement problem with unreliable suppliers that might deliver less than the desired amount. Martínez-de-Albéniz and Simchi-Levi [24] consider the combined use of options and the spot market in inventory control. Haksöz and Seshadri [19] review literature on the use of spot markets in procurement.

The problem we are tackling is similar to the procurement management problem using options studied by Martínez-de-Albéniz and Simchi-Levi [25] and Fu et al. [15, 16], where to meet future demand a buyer could either procure parts from suppliers using fixed price contracts and option contracts (before demand uncertainty is resolved), or tap into the spot market (after demand uncertainty is resolved). Lee et al. [23] address the same problem but they use capacitated options with fixed ordering costs. Chen and Parlar [6] consider an extension of the newsvendor model in which a put option can be purchased to reduce losses resulting from low demand. Capacity reservation contracts are also used for capacity expansion (see [20] and [14], e.g.).

Our article contributes to the literature by combining the behavior of loss-averse customers and the multiple options contract. In general, we present efficient algorithms that deal with piecewise linear utility functions and general demand functions.

3. MODEL SETTING

Let x denote the newsvendor's stochastic demand, following a continuous distribution with cumulative distribution function $F(x)$ and probability density function (PDF) $f(x)$. $F(x)$ is assumed to be continuous, differentiable and invertible. Let $w > 0$ be the wholesale price at which the newsvendor buys from his supplier, and $p > w$ be the newsvendor's retail price. Assume that the unit salvage price s is standardized to zero, and the newsvendor incurs no penalty or goodwill cost but he does lose sales when demand exceeds his inventory. We know that a risk-neutral newsvendor's optimal order quantity Q^* is given by

$$Q^* = F^{-1}\left(\frac{p-w}{p}\right)$$

We characterize the loss aversion behavior of a newsvendor by a constant $\lambda > 1$. That is, when the newsvendor's profit Y is positive, his utility is Y, otherwise his utility is λY. This utility function is widely used to represent loss-averse behavior in economics and management [2, 29, 31].

Now the newsvendor's problem becomes one of deciding how much to order to maximize his expected utility. We could write the newsvendor's expected utility as

$$\lambda \int_0^{wQ/p} (xp - wQ) f(x) dx + \int_{wQ/p}^{Q} (xp - wQ) f(x) dx + (p-w)Q \int_Q^{\infty} f(x) dx$$

The first term is the newsvendor's expected loss (negative gain) when demand is lower than the inventory level, the second term is the newsvendor's expected nonnegative gain when there is enough inventory to meet all demand, and the last term is the newsvendor's expected gain when demand exceeds the inventory level.

It is then not difficult to see that the above utility is concave in Q. Hence, by considering the first-order condition (FOC), the newsvendor's optimal order quantity for a given λ, $Q^*_\lambda(\lambda > 1)$, is given by the following equation

$$pF(Q^*_\lambda) + w - p + (\lambda - 1)wF\left(\frac{wQ^*_\lambda}{p}\right) = 0$$

It is straightforward to show that $Q^* \geq Q^*_{\lambda_1} \geq Q^*_{\lambda_2}$ for any $\lambda_2 \geq \lambda_1 \geq 1$, implying that the newsvendor would prefer to order less to avoid a potential loss. This is consistent with the intuition that the order quantity would decline as loss aversion rises.

Consider now that the newsvendor purchases options from multiple suppliers. Our model is a special case of Martínez-de-Albéniz and Simchi-Levi [25] except that we model loss-aversion in our setting. Basically, there are n options from n different suppliers, with index i denoting the ith one, $i \in \{1, \ldots, n\}$. The reservation price and execution price of the ith option are denoted by (r_i, h_i). Note that $n = 1$ is equivalent to the situation where the newsvendor faces a buy-back contract with wholesale price $r + h$ and buyback credit h. Assume without loss of generality that $r_1 > \cdots > r_n$. We further make the assumption that $r_1 + h_1 < \cdots < r_n + h_n$ (and hence $h_1 < \cdots < h_n$). Although the previous literature (e.g., [15] and [16]) assumes only that $h_1 < \cdots < h_n$, our assumption could be relaxed easily, since if there are two options i and j such that $r_i > r_j$ and $r_i + h_i \geq r_j + h_j$, one would always prefer j to i, hence option i could be removed from the option set. The timeline of events in this problem is as follows (refer to Fig. 1):

- Stage 1: For every option i, $1 \leq i \leq n$, the reservation price and execution cost, (r_i, h_i), are announced.
- Stage 2: The newsvendor decides the reservation quantity q_i of the ith option, where $q_i \geq 0$, and pays the reservation cost $q_i r_i$.
- Stage 3: Demand uncertainty is resolved. Let x denote the newsvendor's revealed demand.
- Stage 4: The newsvendor decides how many options to execute, where the execution quantity of option i is y_i, $0 \leq y_i \leq q_i$, with the restriction $y = \sum_i y_i \leq x$. The newsvendor pays the execution cost $y_i h_i$ for the ith option.
- Stage 5: The newsvendor sells y products at retail price p, and hence collects yp. The unfilled demand $x - y$ is completely lost. Alternatively, assuming the spot price is also p, the excess demand is filled completely through the spot market, which makes no difference for the newsvendor in terms of profit.

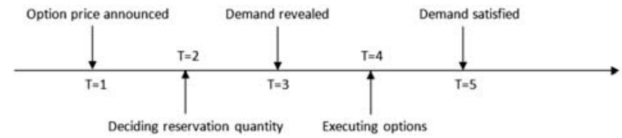

Figure 1. Timeline of events.

The newsvendor's profit is then given by

$$Y = yp - \sum_{i=1}^{n} r_i q_i - \sum_{i=1}^{n} h_i y_i$$

and his utility $U = Y$ when $Y > 0$, and $U = \lambda Y$ otherwise. The larger λ is, the more sensitive the newsvendor to loss. The problem is then to maximize the newsvendor's expected utility.

For convenience, we make the assumption that $r_i + h_i < p$ for every option i (so it follows that $r_1 + h_1 < \cdots < r_n + h_n < p$), otherwise the newsvendor would have no incentive to procure that option and we could simply remove it from the option set. After reservation costs are sunk and demand is revealed, it is optimal for the newsvendor to apply a greedy approach to pick (among the unexecuted) options with the smallest execution cost until all options are executed or demand is completely filled, so his optimal executing strategy is as follows:

- If $\sum_i q_i \leq x$, $y_i = q_i$ for every $1 \leq i \leq n$.
- If $\sum_i q_i > x$, there exists a $1 \leq j \leq n$ such that $y_i = q_i$ when $i < j$, and $y_j = x - \sum_{i=1}^{j-1} q_i$, $y_i = 0$ when $i > j$.

Hence, we could represent the newsvendor's expected profit as

$$\mathbb{E}(Z) = \sum_{i=1}^{n} \left(-r_i q_i + (p - h_i) \int_0^{q_i} \bar{F}\left(\sum_{j=1}^{i-1} q_j + x\right) dx \right)$$

where $\bar{F}(\cdot) = 1 - F(\cdot)$ is the complementary cumulative distribution function of the demand, that is, $\bar{F}(x) = \Pr(\text{Demand} \geq x)$.

We refer to the abovementioned model as the original model. When the newsvendor is risk neutral, that is, $\lambda = 1$, his goal is to maximize the expected profit. This is the same as the procurement problem addressed in Fu et al. [15], which shows that for any two consecutive active options (an option i

is active iff $q_i > 0$) a and b (assume without loss of generality that $a < b$), there is

$$\sum_{i=1}^{a} q_i = F^{-1}\left(1 - \frac{r_a - r_b}{h_b - h_a}\right)$$

In Fu et al. [15], the spot market is also viewed as an option, $r_{n+1} = 0$ and h_{n+1} is the spot price. Hence, our problem can alternatively be described as a procurement problem in which the stochastic demand is x and the newsvendor could either buy options or buy from the spot market ($h_{n+1} = p$). If his reservation quantity is less than the demand, the excess demand is filled through the spot market. If the cost of the combined use of options and the spot market exceeds the cost of using the spot market only, the newsvendor loses, otherwise he gains. The goal is to maximize his expected utility.

In the remaining part of the article, we will use the procurement model to describe our problem. For consistency, p is both the newsvendor's retail price in the newsvendor model and the spot price in the procurement model. To bridge the two models discussed above, it is convenient to say that when the newsvendor has insufficient inventory to serve all demand, he has to make an emergency purchase from the spot market, and the price equals the retail price. So the use of the spot market brings neither risk nor profit to the newsvendor. Although the problem could be generalized to the case where the spot price deviates from the retail price, we restrict our effort to the current model for (conciseness/ease of description) without compromising the main insights from the model.

4. SINGLE OPTION

In this section, we consider the problem where there is only a single option, that is, $n = 1$. For convenience, consider the procurement risk management model where we remove the subscript i and use r, h to denote the reservation and execution costs of the option, respectively, and p to denote the spot price (or the retailing price in the newsvendor model).

4.1. Optimal Solution

Suppose that the newsvendor reserves q units of option, and therefore, his procurement cost when the demand is x is given by

$$C(x) = rq + h\min(x, q)$$

and his revenue is $p\min\{x, q\}$. Our goal is to find a q that maximizes the buyer's expected utility. Note that when $x = rq/(p-h)$, the buyer's utility is zero. Since $r + h < p$, we have $rq/(p-h) < q$. Let f be the PDF of the demand, we could then write the expected utility as a function of q,

$$\begin{aligned} U(q) = \lambda &\int_0^{\frac{rq}{p-h}} (px - hx - rq) f(x)dx \\ &+ \int_{\frac{rq}{p-h}}^{q} (px - hx - rq) f(x)dx \\ &+ \int_q^{\infty} (pq - hq - rq) f(x)dx \end{aligned}$$

Taking the first-order derivative with respect to q gives

$$\frac{\partial U}{\partial q} = (h-p)F(q) - (\lambda - 1)rF\left(\frac{rq}{p-h}\right) - h - r + p$$

It is straightforward to see that U is concave with respect to q, so the optimal order quantity is q^* such that

$$(h-p)F(q^*) - (\lambda - 1)rF\left(\frac{rq^*}{p-h}\right) - h - r + p = 0 \tag{1}$$

Note that $h < p$, $\lambda \geq 1$, the left-hand side of Eq. (1) is continuous and monotonously decreasing in q. Thus, we have shown the uniqueness and existence of the optimal solution q^*. Further, q^* could be found efficiently by performing a binary search.

It should be noted that Wang and Webster [34] consider the loss-averse newsvendor with a single supplier $r > 0, h = 0$, and a goodwill cost, s, is incurred when demand is not filled. We consider the case $h \geq 0$ and no emergency delivery or goodwill cost is incurred.

When loss aversion rises, $\lambda - 1$ increases and the left-hand side of Eq. (1) decreases, given that q^* remains the same. To cover the decrease and keep the equation, the optimal level of q decreases. Thus, as expected, the reservation quantity of the single option decreases when loss aversion rises.

OBSERVATION 1: The buyer would procure less in the loss-averse model than in the risk-neutral model.

When $\lambda = 1$, the buyer is risk-neutral, and the optimal solution is q^* such that

$$F(q^*) = \frac{p-h-r}{p-h}$$

which is consistent with our observation. Conversely, when $F(q) = \frac{p-h-r}{p-h}$, the left-hand side of Eq. (1) larger than zero, hence the optimal solution is less. This implies that a loss-averse buyer would reserve less compared with a risk-neutral buyer with the same demand in the single option model.

OBSERVATION 2: Obviously, when $w = h + r$, the loss-averse buyer would procure more with supply options than in the traditional model. As discussed in Section 3, the optimal ordering quantity for a loss-averse buyer in the traditional model (i.e., the buyer buys products directly) is

$$-pF(q^*) - (\lambda - 1)wF\left(\frac{wq^*}{p}\right) - w + p = 0 \qquad (2)$$

It follows then that when $h = 0$, Eqs. (1) and (2) yield the same solution (in fact the two equations solve the same newsvendor problem). When h increases, the solution to Eq. (2) remains the same, whereas the solution to Eq. (1) decreases. Therefore, the observation follows immediately.

Further, when $w = h + r$, the loss-averse buyer prefers options to firm orders. As shown above, the buyer would order q_2^* units more than q_1^*, the optimal procurement quantity in the traditional model. Hence, $u_2(q_2^*) > u_2(q_1^*)$, where u_2 (resp., u_1) denotes the utility function in the option (resp., traditional) model. It is then easy to see $u_2(q_1^*) > u_1(q_1^*)$, so the observation follows.

4.2. Example

To gain a better understanding of the optimal solution with an option, we consider the following example. Suppose that the newsvendor's demand follows the truncated normal distribution with mean $\mu = 100$ and deviation $\sigma^2 = 100^2$. The demand is restricted to the interval [0, 200], and, to facilitate comparison with firm orders, we consider the options with reservation price r and execution price h such that $r + h = 10$ under different levels of loss aversion, λ, and assume that the retail price is $p = 15$.

We present the optimal reservation strategies with loss aversion varying from $\lambda = 1$ to $\lambda = 5$, and three types of options $(r, h) = (10, 0)$, (8, 2), and (6, 4). Here $h = 0$ corresponds to firm orders. The newsvendor's optimal reservation quantity is given in Table 1, from which we observe the following:

Table 1. Reservation quantities with different risk preferences and options

λ	$(r, h) = (10, 0)$	$(r, h) = (8, 2)$	$(r, h) = (6, 4)$
1	71.0811	80.1247	92.2137
1.5	60.7016	70.0892	83.0159
2	53.0389	62.3794	75.6091
2.5	47.1205	56.2387	69.4753
3	42.4009	51.2160	64.2926
3.5	38.5450	47.0249	59.8454
4	35.3337	43.4713	55.9821
4.5	32.6170	40.4187	52.5916
5	30.2883	37.7671	49.5901

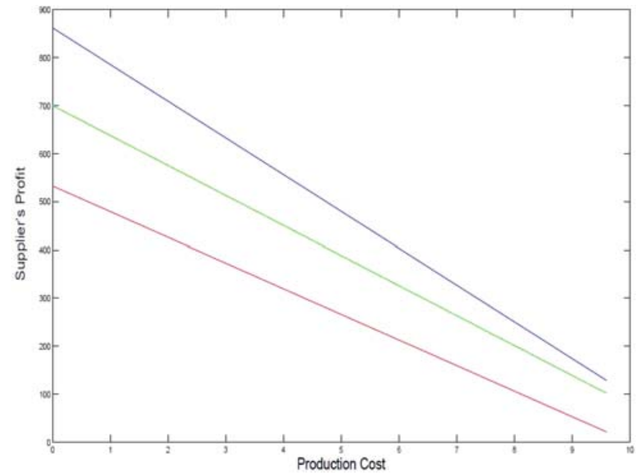

Figure 2. Supplier's profit with different option types and different marginal production costs. The profits when the option parameters (r, h) are equal to (10, 0), (8, 2), and (6, 4) are shown in red, green and blue, respectively. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

- A loss-averse newsvendor would reserve less when loss aversion rises, but this effect is less significant when the option becomes more flexible, that is, with less reservation cost and higher execution price.
- The loss-averse newsvendor tends to reserve more when the option becomes more flexible, and the effect is quite significant even when the newsvendor is strongly loss-averse: although the reservation quantity does not increase that much when loss aversion is strong, the relative increase is much more significant.

One might also wonder if the option contract would benefit a supplier. Suppose that the supplier has constant marginal production cost $c \geq 0$, and if the newsvendor reserves q units from the supplier and executes $y \leq q$ units, the unexecuted $q - y$ units are completely perished without salvage. In Fig. 2, we compute the supplier's expected profit with different options and different marginal production costs, given the newsvendor's loss aversion $\lambda = 2$. It could be seen that $(r, h) = (6, 4)$ benefits the supplier most, and so the introduction of option contracts could be mutually beneficial to both the newsvendor and the supplier. We refer readers to Burnetas and Ritchken [5] and Brown and Lee [4] for discussions of the win–win situation that options could bring.

4.3. A Different Approach

Although the single-option case $n = 1$ could be solved to optimality quite efficiently, the problem becomes much harder in the multioption model where $n > 1$, and it is difficult to generalize the idea adopted above. Let us now consider an alternative approach, which could be easily generalized to more options.

First, let us analyze the change in expected utility, du, when q changes to $q + dq$ for some $|dq| \ll 1$. dq can be either positive or negative. We consider the case where dq is positive. The other case could be analyzed similarly.

- CASE 1: The change in expected utility when the demand is larger than q. Consider the following two subcases:

  CASE 1.1: When the demand is larger than $q + dq$. Since the reservation quantity is q, the additional dq demand has to be realized through the spot market in the optimal solution, but through option in the new solution. Hence, the change in utility is

$$\begin{aligned}\Delta_{11} &= (p-h-r)(1-F(q+dq))dq\\ &= (p-h-r)(1-F(q)\\ &\quad - F'(q+\xi dq)dq)dq\\ &= (p-h-r)(1-F(q))dq + o(dq)\end{aligned}$$

  for some $\xi \in (0, 1)$.

  CASE 1.2: When the demand is between q and $q + dq$, we have

$$\begin{aligned}\Delta_{12} &\le (p-h-r)(F(q+dq)-F(q))dq\\ &= (p-h-r)F'(q+\gamma dq)dq^2\end{aligned}$$

  for some $\gamma \in (0, 1)$. This is because Δ_{12} reaches its maximum when all additional dq options are executed and reaches its minimum when none of the additional dq options are executed,

$$\Delta_{12} \ge -rF(q+\gamma dq)dq^2$$

  Since dq^2 is negligible relative to dq, that is, $dq^2 = o(dq)$, we immediately arrive at the following result:

$$\begin{aligned}\Delta_1 &= \Delta_{11} + \Delta_{12}\\ &= (p-h-r)(1-F(q))dq + o(dq)\end{aligned}$$

- CASE 2: The change in expected utility when the demand is between $rq/(p - h)$ and q, which is given by

$$\Delta_2 = -r\left(F(q) - F\left(\frac{rq}{p-h}\right)\right)dq + o(dq)$$

- CASE 3: The change in expected utility when the demand is below $rq/(p - h)$,

$$\Delta_3 = -r\lambda F\left(\frac{rq}{p-h}\right)dq + o(dq),$$

and therefore, $du = \Delta_1 + \Delta_2 + \Delta_3$. It can further be shown that, when dq is negative, the result will still hold. We omit the analysis here. Then,

$$\begin{aligned}du &= (p-h-r)(1-F(q))dq\\ &\quad - r\left(F(q) - F\left(\frac{rq}{p-h}\right)\right)dq\\ &\quad - r\lambda F\left(\frac{rq}{p-h}\right)dq + o(dq)\\ &= \Bigg((p-h-r) + (h-p)F(q)\\ &\quad + (1-\lambda)rF\left(\frac{rq}{p-h}\right)\Bigg)dq + o(dq)\end{aligned}$$

It follows then that

$$\begin{aligned}\frac{du}{dq} &= (p-h-r) + (h-p)F(q)\\ &\quad + (1-\lambda)rF\left(\frac{rq}{p-h}\right)\end{aligned}$$

By equating $\frac{du}{dq}$ to zero, we maximize the expected utility. Not surprisingly, the optimal solution is exactly what Eq. (1) yields. Hence, our analysis in this subsection is consistent with the previous approach.

5. *n* OPTIONS

In this section, we consider the general problem where there are n options. Again, denote the reservation price of the ith option by r_i and the execution price by h_i, and assume without loss of generality that $r_1 > \cdots > r_n, r_1+h_1 < \cdots < r_n + h_n$. Let p be the retail price (or the spot price).

5.1. The Existence of an Optimal Solution

In this subsection, we show that an optimal solution exists. Let $D_{\max}$ be the maximum demand, and pick an $M \ge D_{\max}$. We consider the solution where the procurement quantity of each option takes a value from $[0, M]^n$, that is, $q_i \in [0, M]$ for every $1 \le i \le n$. Let $u(\mathbf{q})$ be the expected utility corresponding to procurement strategy $\mathbf{q} = (q_1, \ldots, q_n)$. One could immediately see that u is continuous in $\mathbf{q}$, and since $[0, M]^n$ is a compact set, by the Weierstrass theorem, there exists an optimal solution.

Conversely, when $q_i > M$ for some i, because M is too large and precious capital is wasted on reserving unnecessary options, one expects $u(\mathbf{q}) < u(\mathbf{q}')$, where $q' = (q_1, \ldots, q_{i-1}, D_{\max}, q_{i+1}, \ldots, q_n)$. $\mathbf{q}$ is, of course, not the optimal reservation strategy. Thus, we have shown the existence of an optimal solution. From here onward, let $\mathbf{q} = (q_1, \ldots, q_n)$ denote the optimal procurement strategy.

5.2. Subproblems

For an optimal procurement strategy $\mathbf{q}$, there exists a $d_b \geq 0$ such that when the demand $d = d_b$, the newsvendor's utility is zero. As we have shown in Section 3, after demand is revealed, the newsvendor executes the options following a greedy approach, hence we could compute the newsvendor's utility easily. The uniqueness of d_b is also quite straightforward, and there exists a $0 \leq k < n$ such that

$$Q_k < d_b \leq Q_{k+1}$$

and

$$pd_b = \sum_{i=1}^{n} r_i q_i + \sum_{i=1}^{k} h_i q_i + (d_b - Q_k)h_{k+1} \qquad (3)$$

where $Q_k = \sum_{i=1}^{k} q_i$ is the total reservation quantity of the first k options, $q_k = Q_k - Q_{k-1}$.

The subproblem P_k is defined as the original n-option reservation problem with the additional constraint $Q_k < d_b \leq Q_{k+1}$. Intuitively, in subproblem P_k, if the utility is zero, one could conclude that options 1 to k are executed, and options $k + 2$ to n are not executed, and option $k + 1$ is completely or partially executed. It follows immediately that the optimal solution falls to be one of the subproblems, hence if we could solve subproblem P_k for every $0 \leq k < n$, the whole problem could be solved.

5.3. Characterizing Subproblem P_k

In this subsection, we assume that all options are active in the optimal solution, and this assumption is relaxed later in Section 6. Consider two consecutive active options i and $i + 1$. If a strategy $\mathbf{q} = (q_1, \ldots, q_n)$ is optimal for the subproblem, then we consider the alternative strategy $\mathbf{q}' = (q_1, \ldots, q_i - dq, q_{i+1} + dq, \ldots, q_n)$ for some $|dq| \ll 1$. Note that dq can be either positive or negative. Again, we are trying to study the change in expected utility given the value of du. We adopt the methodology used in Subsection 4.1 with slight modification. Consider the following two cases:

CASE 1: When $i \leq k$, $du = u(\mathbf{q}') - u(\mathbf{q})$ consists of the following parts. Similar to the analysis in Section 4, one could immediately come up with,

- The change in expected utility when the demand is larger than d_b is

$$\Delta_1 = (r_i + h_i - r_{i+1} - h_{i+1})(1 - F(d_b))dq + o(dq)$$

- The change in expected utility when the demand is between $Q_i = \sum_{j=1}^{i} q_j$ and d_b is

$$\Delta_2 = \lambda(r_i + h_i - r_{i+1} - h_{i+1})(F(d_b) - F(Q_i))dq + o(dq)$$

- The change in expected utility when the demand is smaller than Q_i is

$$\Delta_3 = \lambda(r_i - r_{i+1})F(Q_i)dq + o(dq)$$

It follows that $du = \Delta_1 + \Delta_2 + \Delta_3$, and supposing that $\mathbf{q}$ is the optimal solution, we have $du \to 0$ when $dq \to 0$. Then we are able to formulate an equation that is linear in $F(Q_i)$ and $F(d_b)$,

$$(r_i + h_i - r_{i+1} - h_{i+1}) + (\lambda - 1)(r_i + h_i - r_{i+1} - h_{i+1})F(d_b) + \lambda(h_{i+1} - h_i)F(Q_i) = 0$$

and for ease of description, we refer to these k equations as $g_i(F(Q_i), F(d_b)) = 0, i \leq k$.

Given that i and $i + 1$ are two consecutive active options, the loss-averse newsvendor would set his Q_i smaller than a risk-neutral newsvendor would, which is consistent with one's intuition that a loss-averse newsvendor would tend to purchase less. However, as we will show later, the set of active options might change when loss aversion rises, and the result will not necessarily hold when i and $i + 1$ are not active in the risk-neutral settings.

CASE 2: When $i > k$, $du = u(\mathbf{q}') - u(\mathbf{q})$ consists of the following parts:

- The change in utility when the demand is larger than Q_i is

$$\Delta_1 = (r_i + h_i - r_{i+1} - h_{i+1})(1 - F(Q_i))dq + o(dq)$$

- The change in utility when the demand is between d_b and Q_i is

$$\Delta_2 = (r_i - r_{i+1})(F(Q_i) - F(d_b))dq + o(dq)$$

- The change in utility when the demand is smaller than d_b is

$$\Delta_3 = \lambda(r_i - r_{i+1})F(d_b)dq + o(dq)$$

Again, we have $du \to 0$ when $dq \to 0$, hence

$$\begin{aligned}&(r_i + h_i - r_{i+1} - h_{i+1})\\&\quad + (\lambda - 1)(r_i - r_{i+1})F(d_b)\\&\quad + (h_{i+1} - h_i)F(Q_i) = 0\end{aligned}$$

which is referred to as $g_i(F(Q_i), F(d_b)) = 0, i > k$.

Similarly, given that i and $i + 1$ are two consecutive active options, the newsvendor would decrease his Q_i when he changes from being risk-neutral to being loss-averse.

For completeness, let $r_{n+1} = 0$ and $h_{n+1} = p$, where the approach described above still applies. Now we have $n + 1$ equations, $g_i(F(Q_i), F(d_b)) = 0$ for every $1 \le i \le n$, and the determination of d_b is as follows:

$$pd_b = \sum_{i=1}^{n} r_i q_i + \sum_{i=1}^{k} h_i q_i + (d_b - Q_k)h_{k+1}$$

We can compute the optimal solution based on these observations.

5.4. Approximating the Optimal Solution

In the previous subsection, we characterized $n + 1$ equations to describe the optimal solution. However, these equations are very difficult to solve when the demand follows certain distributions, for example, the truncated normal distribution and the exponential distribution.

As analyzed above, g_i is linear in $F(Q_i)$ and $F(d_b)$. Thus, for simplicity, we represent g_i as

$$g_i = \alpha_i F(Q_i) + \beta_i F(d_b) + \gamma_i$$

Now consider the linear interpolation of the cumulative distribution function of the demand F. Suppose that the cumulative distribution of the demand has t pieces, while the ith piece is defined over $I_i = [a_{i-1}, a_i]$ for any $1 \le i \le t$. $a_0 < \cdots < a_n$ are the folding points of F.

A naive approach to the problem is to enumerate $1 \le k_1 \le \cdots \le k_n \le t$ and $1 \le k \le t$ such that $Q_i \in I_{k_i}$ and $d_b \in I_k$, therefore, each of the equations $g_i = 0$ becomes linear and, together with Eq. (3), we would have $n + 1$ linear equations over $n + 1$ variables, $Q_1, \ldots, Q_n$ and d_b, so the problem becomes solvable. However, enumerating $1 \le k_1 \le \cdots \le k_n \le t$ takes exponential time and would thus be inefficient.

Now consider a slightly different approach. Note that for each equation

$$\alpha_i F(Q_i) + \beta_i F(d_b) + \gamma_i = 0$$

The corresponding d_b increases or decreases monotonously in Q_i. For every $1 \le i \le n, 1 \le j \le t$, let a_{ij} be the solution of d_b in equation $g_i = 0$, given that $Q_i = a_j$. It follows that when $d_b \in [\min(a_{i,j-1}, a_{i,j}), \max(a_{i,j-1}, a_{i,j})]$, $Q_i \in I_j$. Let

$$A = \left\{a_{ij} | 1 \le i \le n, 0 \le j \le t\right\} \cup \left\{a_j | 0 \le j \le t\right\}$$

and we sort the elements of A and rename them as $e_1, \ldots, e_{|A|}$ such that $e_1 < \cdots < e_{|A|}$. It follows that within each interval $[e_i, e_{i+1}]$, for any Q_j, there must exist an I_u such that $Q_j \in I_u$. Otherwise, assume that Q_j could take any value from I_u and I_{u+1}, and we have $a_{ju} \in [e_i, e_{i+1}]$, which contradicts the assumption that e_i, e_{i+1} are two consecutive elements in A.

Then it suffices to consider $d_b \in [e_{i-1}, e_i]$ for each $1 \le i \le |A|$, where the value of $F(Q_i)$ is linear in Q_i. Thus, we have $n + 1$ linear functions, and thus, solvable. Note that $|A| \le (t + 1)(n + 1)$, so the entire problem is solvable in polynomial time of t and n.

6. SELECTING ACTIVE OPTIONS

In the previous section, we found a way to solve the newsvendor problem in polynomial time given the set of active options, that is, the options with nonzero reservation quantities. However, in reality, we do not know whether or not an option is active.

A straightforward solution is to enumerate the active options. Assuming that there are n options available, there are 2^n possible combinations of active options, which is exponentially large, and thus, impractical.

6.1. When d_b is Known

In this subsection, we analyze the properties of active options when d_b is known. Further, assume that $Q_k < d_b \le Q_{k+1}$. In this way, we know that option $k + 1$ is active since $q_{k+1} = Q_{k+1} - Q_k > 0$. Let $i_1 < \cdots < i_{t+1}, i_{t+1} = k + 1$ denote the set of active options before $k + 1$. Following the analysis in Section 5, we have

$$\begin{aligned}&(h_{i_{j+1}} - h_{i_j})F(Q_{i_j}) + (\lambda - 1)(r_{i_j} - r_{i_{j+1}})F(d_b)\\&\quad = r_{i_{j+1}} + h_{i_{j+1}} - r_{i_j} - h_{i_j}\end{aligned}$$

Therefore, for every $j \le t$, when d_b is known, we can compute Q_{i_j} assuming i_j, i_{j+1} are two consecutive active options. Let $Q_a(b)$ be the value of Q_a when $a < b$ are two consecutive active options.

PROPOSITION 1: If $k + 1 > 1$,

$$i_t = \text{argmax}_x \left\{Q_x(k + 1)\right\}^1$$

[1]It is possible that $\text{argmax}_x\{Q_x(k + 1)\}$ is not a singleton. Then, it suffices to pick the smallest one from the correspondence set $\text{argmax}_x\{Q_x(k + 1)\}$ as i_t.

PROOF: The intuition here is that we should reserve as many units as possible before $k + 1$. For a given solution with $\text{argmax}_x \{Q_x(k+1)\}$ not used, we can always improve it by reserve at that point as well.

The proof is divided into two parts. In the first part, we will show that, $z = \text{argmax}_x \{Q_x(k+1)\}$ is an active option in subproblem P_k, and in the second part, we will show that $i_t = z$. Now let us begin with the first part.

Suppose for contradiction that z is not an active option in the optimal strategy, we immediately have

$$Q_{i_t}(k+1) < Q_z(k+1)$$

Denote the optimal solution by $\mathbf{q} = (q_1, \ldots, q_n)$ such that $q_{k+1} > 0$ and $q_z = 0$. Then, consider the alternative solution $\mathbf{q}' = (q_1, \ldots, q_z + dq, \ldots, q_{k+1} - dq, \ldots, q_n)$ for some $0 < dq \ll 1$.

We could come up with

$$\begin{aligned} u(\mathbf{q}') - u(\mathbf{q}) \geq\ & (r_{k+1} + h_{k+1} - r_z - h_z)(1 - F(d_b))dq \\ & + \lambda(r_{k+1} + h_{k+1} - r_z - h_z)(F(d_b) \\ & - F(Q_k))dq \\ & + \lambda(r_{k+1} - r_z)F(Q_k)dq + o(dq) \end{aligned}$$

Let us explain the inequality above. When demand $D > Q_k$, we satisfy the next dq units of demand using option $k + 1$ in the optimal solution $\mathbf{q}$, while we use option z in $\mathbf{q}'$. Conversely, we have

$$\begin{aligned} & (r_{k+1} + h_{k+1} - r_z - h_z)(1 - F(d_b)) \\ & \quad + \lambda(r_{k+1} + h_{k+1} - r_z - h_z)(F(d_b) - F(Q_z(k+1))) \\ & \quad + \lambda(r_{k+1} - r_z)F(Q_z(k+1)) = 0 \end{aligned}$$

and since $Q_{i_t}(k+1) < Q_z(k+1)$, we have

$$\begin{aligned} u(\mathbf{q}') - u(\mathbf{q}) \geq\ & \lambda(h_{k+1} - h_z)(F(Q_z(k+1)) \\ & - F(Q_k))dq + o(dq) > 0 \end{aligned}$$

Hence $u(\mathbf{q}') > u(\mathbf{q})$, contradicting the optimality of $\mathbf{q}$, which gives $q_z > 0$. This completes the first part of our proof.

As for the second part, suppose for contradiction that z is an active option, but not the last active option before $k + 1$, that is, $i_t > z$. Based on the approach in the first part, we know that $Q_z \geq Q_z(k+1)$, and, as i_t is the last active option before $k+1$, we have $Q_{i_t} = Q_{i_t}(k+1)$. This gives $Q_{i_t} < Q_z$, that is, $q_{z+1} + \cdots + q_{i_t} < 0$, contradicting the feasibility, thus completes our proof. □

Similar to the proof of Proposition 1, we have the following proposition:

PROPOSITION 2: For any $1 \leq j \leq t$, if $i_{j+1} > 1$,

$$i_j = \text{argmax}_{x < i_{j+1}} \left\{Q_x(i_{j+1})\right\}$$

Based on Propositions 1 and 2, we could determine all active options, $i_1, \ldots, i_t$, before $k + 1$ in polynomial time.

Now let us consider all active options following $k + 1$, denoted by $k + 1 = u_0 < u_1 < \cdots < u_{s+1} = n + 1$. Similar to Proposition 1, we have the following Proposition,

PROPOSITION 3:

$$u_1 = \text{argmin}_x \{Q_{k+1}(x)\}$$

PROOF: Again, the proof is divided into two parts. In the first part, we will show that, $z = \text{argmin}_x \{Q_{k+1}(x)\}$ is an active option in subproblem P_k, and in the second part, we will that $u_1 = z$. Now let us begin with the first part.

Suppose for contradiction that z is not an active option in the optimal strategy, we immediately have

$$Q_{k+1}(u_1) > Q_{k+1}(z)$$

Denote the optimal solution by $\mathbf{q} = (q_1, \ldots, q_n)$ such that $q_{k+1} > 0$ and $q_z = 0$. Then, consider the alternative solution $\mathbf{q}' = (q_1, \ldots, q_{k+1} - dq, \ldots, q_z + dq, \ldots, q_n)$ for some $0 < dq \ll 1$.

We could immediately come up with

$$\begin{aligned} u(\mathbf{q}') - u(\mathbf{q}) \geq\ & (r_{k+1} + h_{k+1} - r_z - h_z)(1 - F(Q_{k+1}))dq \\ & + (r_{k+1} - r_z)(F(Q_{k+1}) - F(d_b))dq \\ & + \lambda(r_{k+1} - r_z)F(d_b)dq + o(dq) \end{aligned}$$

The intuition is, when the demand $D > Q_{k+1} - dq$, the next dq units of demand is filled using option $k + 1$ in optimal solution $\mathbf{q}$, and filled using option z in $\mathbf{q}'$. Conversely, we have

$$\begin{aligned} & (r_{k+1} + h_{k+1} - r_z - h_z)(1 - F(Q_{k+1}(z))) \\ & \quad + (r_{k+1} - r_z)(F(Q_{k+1}(z)) - F(d_b)) \\ & \quad + \lambda(r_{k+1} - r_z)F(d_b) = 0 \end{aligned}$$

and since $Q_{k+1} = Q_{k+1}(u_1) > Q_{k+1}(z)$, we have

$$\begin{aligned} u(\mathbf{q}') - u(\mathbf{q}) \geq\ & (h_z - h_{k+1})(F(Q_{k+1}) \\ & - F(Q_{k+1}(z)))dq + o(dq) > 0 \end{aligned}$$

Hence $u(\mathbf{q}') > u(\mathbf{q})$, and so $q_z > 0$.

The second part of the proof is similar to the proof of Proposition 1, and thus we omit it here. □

PROPOSITION 4: For any $1 \leq j \leq s$, if $i_{j-1} < n+1$,

$$u_j = \mathrm{argmin}_{x > i_{j-1}} \left\{ Q_{u_{j-1}}(x) \right\}$$

In this way, when d_b is known, we could "catch" all active options for each subproblem P_k, and hence the problem is solvable.

6.2. d_b is Unknown

In this subsection, we focus on solving subproblem P_k when d_b is unknown. Following the analysis in the previous subsection, we need to compare $Q_i(j)$ and pick up the maximizer or minimizer for a given i or j. In the previous subsection, this was done by computing $Q_i(j)$ for every $1 \leq i < j \leq k+1$ and $k+1 \leq i < j \leq n+1$, but this is impossible when d_b is unknown.

PROPOSITION 5: For every $1 \leq i_1 < i_2 < j \leq k+1$, there exists a $d(i_1, i_2, j)$ such that either

- $d_b > d(i_1, i_2, j)$ implies $Q_{i_1}(j) > Q_{i_2}(j)$, and $d_b < d(i_1, i_2, j)$ implies $Q_{i_1}(j) < Q_{i_2}(j)$ or
- $d_b > d(i_1, i_2, j)$ implies $Q_{i_1}(j) < Q_{i_2}(j)$, and $d_b < d(i_1, i_2, j)$ implies $Q_{i_1}(j) > Q_{i_2}(j)$

PROOF: First, when i and j are two consecutive active options, we have

$$(r_i + h_i - r_j - h_j) + (\lambda - 1)(r_i + h_i - r_j - h_j)F(d_b) + \lambda(h_{i+1} - h_i)F(Q_i) = 0 \tag{4}$$

Therefore, there exist $\alpha_{i_1,j}, \alpha_{i_2,j}, \beta_{i_1,j}, \beta_{i_2,j}, \gamma_{i_1,j}$, and $\gamma_{i_2,j}$ such that

$$\alpha_{i_1,j} F(Q_{i_1}(j)) + \beta_{i_1,j} F(d_b) + \gamma_{i_1,j} = 0,$$
$$\alpha_{i_2,j} F(Q_{i_2}(j)) + \beta_{i_2,j} F(d_b) + \gamma_{i_2,j} = 0$$

Now let $F(Q_{i_1}(j)) = F(Q_{i_2}(j))$. The above two equations, if not linearly dependent, could give a unique $F(d_b)$, and denote the corresponding d_b by $d(i_1, i_2, j)$. Otherwise, $d(i_1, i_2, j) = 0$. □

By Proposition 5, when $d(i_1, i_2, j)$ is known, one could compare $Q_{i_1}(j)$ and $Q_{i_2}(j)$ easily. Similarly, we have the following proposition,

PROPOSITION 6: For every $k+1 \leq i < j_1 < j_2 \leq n+1$, there exists a $d(i, j_1, j_2)$ such that either

- $d_b > d(i, j_1, j_2)$ implies $Q_i(j_1) > Q_i(j_2)$, and $d_b < d(i, j_1, j_2)$ implies $Q_i(j_1) < Q_i(j_2)$, or
- $d_b > d(i, j_1, j_2)$ implies $Q_i(j_1) < Q_i(j_2)$, and $d_b < d(i, j_1, j_2)$ implies $Q_i(j_1) > Q_i(j_2)$

Now, let us call the corresponding $d(i, j, k)$ cutoff points. There are $O(n^3)$ cutoff points, and so there are $O(n^3)$ intervals, such that when d_b is within some interval, it is possible to compare $Q_i(j)$ when i or j is fixed, and so the operations performed in the previous section apply here. Thus, the global problem is solvable in polynomial time.

7. GENERAL DISTRIBUTIONS

In Section 5, we solved the problem in polynomial time by taking the linear interpolation of the cumulative demand function. This works well sometimes, but for many distributions, like the truncated normal distribution or the exponential distribution, it is not always easy to interpolate the cumulative demand distribution, or the interpolated function may have too many pieces, which makes the computation inefficient.

7.1. Solution Procedure

As we have shown in Section 6, for any subproblem P_k can be further divided into a polynomial number of subproblems, with the set of active options fixed (the effort we made in Section 6 also applies to the general distributions). Hence it suffices to solve the problem with the active options known. For ease of illustration, we consider only the subcase of P_k where all options are active, while d_b is restricted to some interval $[\underline{d}, \bar{d}]$.

Following the analysis in Section 5, if we know that $d_b = y$ for some $y \in [\underline{d}, \bar{d}]$, we could compute Q_i for any $1 \leq i \leq n$ by the following equations: when $i \leq k$,

$$(r_i + h_i - r_{i+1} - h_{i+1}) + (\lambda - 1)(r_i + h_i - r_{i+1} - h_{i+1}) \times F(d_b) + \lambda(h_{i+1} - h_i)F(Q_i) = 0 \tag{5}$$

and for $i > k$, we have

$$(r_i + h_i - r_{i+1} - h_{i+1}) + (\lambda - 1)(r_i - r_{i+1})F(d_b) + (h_{i+1} - h_i)F(Q_i) = 0 \tag{6}$$

Then, after replacing $F(d_b)$ with $F(y)$, we could compute Q_i for any i by performing a binary search. Therefore, we could solve the problem since $q_i = Q_i - Q_{i-1}$.

The problem is that we do not know the value of d_b before all Q_i's are known. One possible way to overcome this is to guess a $y \in [\underline{d}, \bar{d}]$, and compute Q_i's under the assumption that $d_b = y$. To distinguish it from the real Q_i, we denote the computed value by $Q_i(y)$. Then, with $Q_i(y)$ known, we could compute the corresponding d_b, denoted by $d_b(y)$, using the equation

$$pd_b(y) = \sum_{i=1}^{n} r_i q_i(y) + \sum_{i=1}^{k} h_i q_i(y) + (d_b(y) - Q_k(y))h_{k+1} \tag{7}$$

where $q_i(y) = Q_i(y) - Q_{i-1}(y)$. If the computed d_b coincides with y, that is, $d_b(y) = y$, then our guess is correct.

Unfortunately, it is almost impossible to guess d_b correctly. Nevertheless, based on the returned result $d_b(y)$, we could adjust our guess.

LEMMA 1: $\forall y_1, y_2 \in [\underline{d}, \bar{d}]$, if $y_1 \leq y_2$, then $d_b(y_1) \geq d_b(y_2)$.

PROOF: First, we would show that $Q_i(y_1) \leq Q_i(y_2)$ when $i \leq k$, and $Q_i(y_1) \geq Q_i(y_2)$ when $i > k$. This is because in Eq. (4), $r_i + h_i < r_{i+1} + h_{i+1}$, and $h_{i+1} > h_i$; as for Eq. (5), we have $r_i > r_{i+1}$ and $h_{i+1} > h_i$, which gives that $Q_i(y_1) \geq Q_i(y_2)$. By Eq. (7), we have

$$
\begin{aligned}
(p - h_{k+1})d_b(y) &= \sum_{i=1}^{k}(r_i + h_i)q_i(y) \\
&\quad + \sum_{i=k+1}^{n} r_i q_i(y) - h_{k+1}Q_k(y) \\
&= \sum_{i=1}^{k-1}(r_i + h_i - r_{i+1} - h_{i+1})Q_i(y) \\
&\quad + \sum_{i=k+1}^{n-1}(r_{i+1} - r_i)Q_i(y) \\
&\quad + (r_k + h_k - r_{k+1} - h_{k+1})Q_k(y) + r_n Q_n
\end{aligned}
$$

It follows that $d_b(y_2) \leq d_b(y_1)$, which completes our proof. □

Now it is possible for us to perform a binary search of d_b. Beginning with $l = \underline{d}$ and $h = \bar{d}$, we compute $y = (l+h)/2$, and the corresponding $d_b(y)$. If $d_b(y) > y$, set $l \leftarrow (l+h)/2$, otherwise set $h \leftarrow (l+h)/2$, and begin a new round of search.

7.2. Extension to Piecewise Linear Utility Functions

In the above analysis, we assumed that the utility function has only one kink point, which is zero. To capture generality, we shall consider the extension of the basic model to piecewise linear functions with more than one kink point. Although firms do not necessarily use piecewise linear utility functions to make business decisions, this setup is quite convenient for risk-neutral firms in a taxing context (where the firm's utility is its pretax profit minus the tax). For example, in the UK, from April 1, 2013 onward, the corporate tax rate is 20% for profits of £300,000 or below and 23% for profits above. This implies a piecewise linear function with two kink points, one at 0 and another at 300,000. In Greece, the tax rate is 26% for profits of 50,000 euros or above, and 33% for profits above. A similar tax structure is also used in South Korea, Germany, and many other countries. Under the new setup, the key task is to compute the demand values $d_b^1, d_b^2, \ldots$ that map the profits to the kink points.

The general approach cannot be solved efficiently, unless there is only a constant number (i.e., $O(1)$) of kink points. For ease of exposition and without compromising the main insights from the model, we consider the case with two kink points.

First, we define subproblem $P_{x,y}$ such that

$$
\begin{aligned}
&Q_x < d_b^1 \leq Q_{x+1},\ Q_y < d_b^2 \leq Q_{y+1} \\
&q d_b^1 - \left(\sum_{i=1}^{n} r_i q_i + \sum_{i=1}^{x} h_i q_i + (d_b^1 - Q_x)h_{x+1}\right) = \pi_1 \\
&q d_b^2 - \left(\sum_{i=1}^{n} r_i q_i + \sum_{i=1}^{y} h_i q_i + (d_b^2 - Q_y)h_{y+1}\right) = \pi_2
\end{aligned}
$$

where π_1 and π_2 are the utilities corresponding to the two kink points. Note that the number of subproblems in polynomial in $O(1)$, thus it suffices to show that each subproblem can be solved efficiently.

For a given subproblem, we solve it through the following procedure:

- Step 1. Make an initial guess of $d_b^2 \in [\underline{d}, \bar{d}]$.
- Step 2. Conditional on the value of d_b^2, compute the value of d_b^1 using binary search until convergence. Let $d_b^1(d_b^2)$ be the obtained value of d_b^1 conditional on d_b^2.
- Step 3. Conditional on the value $d_1^b(d_b^2)$, compute the value of d_b^2 using binary search and let $d_b^2(d_b^1(d_b^2))$ denote that value.
- Step 4. If the initial guess is correct, we have $d_b^2(d_b^1(d_b^2)) = d_b^2$, and the algorithm completes. If this does not hold, we update the initial guess and return to Step 2. The value updating process is again a binary search.

A more detailed analysis follows. Suppose that when profit $\pi \leq \pi_1$, the utility is $\lambda_1\pi + c_1$; when $\pi_1 < \pi \leq \pi_2$, the utility is $\lambda_2\pi + c_2$; and when $\pi_2 < \pi$, the utility is $\lambda_3\pi + c_3$, where λ_3 is normalized to 1 without loss of generality.

Similar to the approach described in Subsection 7.1, we consider the case where all options are active. Then, analogous to Eqs. (4) and (5), we have:

- When $i \leq x$,

$$
\begin{aligned}
&(r_i + h_i - r_{i+1} - h_{i+1}) + (\lambda_1 - 1)(r_i + h_i - r_{i+1} \\
&\quad - h_{i+1})F(d_b^1) \\
&\quad + (\lambda_2 - 1)(r_i + h_i - r_{i+1} - h_{i+1})(F(d_b^2) \\
&\quad - F(d_b^1)) + \lambda_1(h_{i+1} - h_i)F(Q_i) = 0
\end{aligned}
$$

- When $x < i \le y$, we have

$$(r_i + h_i - r_{i+1} - h_{i+1}) + (\lambda_1 - 1)(r_i + h_i - r_{i+1} - h_{i+1})F(d_b^1) + (\lambda_2 - 1)(r_i + h_i - r_{i+1} - h_{i+1})(F(d_b^2) - F(d_b^1)) + \lambda_2(h_{i+1} - h_i)F(Q_i) = 0$$

- When $i > y$, we have

$$(r_i + h_i - r_{i+1} - h_{i+1}) + (\lambda_1 - 1)(r_i + h_i - r_{i+1} - h_{i+1})F(d_b^1) + (\lambda_2 - 1)(r_i + h_i - r_{i+1} - h_{i+1})(F(d_b^2) - F(d_b^1)) + (h_{i+1} - h_i)F(Q_i) = 0$$

Therefore, when d_b^1 and d_b^2 are known, we can compute the reservation quantity $q_i = Q_i - Q_{i-1}$ of option i directly. Now suppose that we already know the value of d_b^2, and we make an initial guess of $d_b^1 = z$. We could then compute, conditional on the guess z, the related values $q_i(z)$ for each option i. Then, using the following equation we can compute $d_b^1(z)$:

$$pd_b^1(z) - \pi_1 = \sum_{i=1}^{n} r_i q_i(z) + \sum_{i=1}^{x} h_i + (d_b^1 - Q_x(z))h_{x+1} \tag{8}$$

If $d_b^1(z) = d_b^1 = z$. Then our guess of d_b^1 is correct and the problem is solved. Otherwise, we can adjust our prior guess according to the relationship between $d_b^1(z)$ and d_b^1. By performing a binary search, we will eventually reach the correct value of d_b^1. This shows how Steps 2 and 3 are executed in our algorithm.

We do not know the value of d_b^2 either. But similarly, we make an initial guess of $d_b^2 = z$, and conditional on the prior belief and using the approach above, we can obtain the value of d_b^1 conditional on $d_2 = z$, denoted by $d_b^1(d_b^2 = z)$. Now, we can compute the quantity of each option. Then, using

$$pd_b^2 - \pi_1 = \sum_{i=1}^{n} r_i q_i + \sum_{i=1}^{x} h_i + (d_b^2 - Q_y(z))h_{y+1} \tag{9}$$

We obtain $d_b^2(d_b^2 = z)$: the returned value of d_b^2 conditional on the prior belief $d_b^2 = z$. If $d_b^2(d_b^2 = z) = z$, our prior belief is correct and we are done. Otherwise, we correct our prior belief and perform a binary search until $d_b^2(d_b^2 = z) = z$ is satisfied. This completes our algorithm.

Note that not the options are not all necessarily active. As Section 6 has already shown us how to pick up the active options, this problem can be solved. Hence our algorithm can be implemented.

It is easy to see that the whole process terminates in polynomial time. When the number of kink points increases, we can simply extend the algorithm by adding outside loops of binary search. The algorithm remains efficient when the number of kink points is $O(1)$.

Consider now a simple case with only one supply option. Suppose that the reservation quantity is Q, then the seller's profit is

$$\pi = (p - h)x - rQ$$

when demand $x \le Q$ and $\pi = (p - h)Q - rQ$ otherwise. Then, the seller's expected utility can be written as

$$\int_0^{\frac{\pi_1 + rQ}{p-h}} (\lambda_1((p - h)x - rQ) + c_1)f(x)dx + \int_{\frac{\pi_1 + rQ}{p-h}}^{\frac{\pi_2 + rQ}{p-h}} (\lambda_2((p - h)x - rQ) + c_2)f(x)dx + \int_{\frac{\pi_2 + rQ}{p-h}}^{Q} (\lambda_3((p - h)x - rQ) + c_3)f(x)dx + \int_Q^{+\infty} (\lambda_3(p - h - r)Q + c_3)f(x)dx$$

The FOC yields Q^*, the optimal value of Q:

$$\lambda_3(p - h)F(Q^*) + r(\lambda_2 - \lambda_3)F\left(\frac{\pi_2 + rQ}{p - h}\right) + r(\lambda_1 - \lambda_2)F\left(\frac{\pi_1 + rQ}{p - h}\right) = \lambda_3(p - h - r)$$

When the utility function is concave, $\lambda_1 \ge \lambda_2 \ge \lambda_3$ and the left-hand side of the equation is increasing in Q^*. To keep the results meaningful, normalize the parameters so that $\lambda_1 + \lambda_2 + \lambda_3 = 1$. Then, an increase in λ_1 is accompanied by a decrease in λ_2 and λ_3, which lowers Q^*. Similarly, an increase in λ_3 pushes Q^* higher.

7.3. Example

Consider an example where, similar to the example given in Section 4, the newsvendor's demand follows the truncated normal distribution $N(100, 200^2)$ defined over [0,200]. There are five options, given in Table 2. The newsvendor's selling price is 20.

Table 2. Reservation and execution prices of options

	1	2	3	4	5
Reservation price r_i	10	8	6	4	2
Execution cost h_i	4.5	6.8	9.5	12.6	16.1

Table 3. Reservation strategies with different risk preferences

	$\lambda = 1.0$	$\lambda = 1.2$	$\lambda = 1.5$	$\lambda = 2$	$\lambda = 3$
q_1	31.8260	27.4146	22.9174	17.6489	13.0255
q_2	25.7372	22.5184	19.2116	15.2129	10.0791
q_3	17.3349	23.3601	28.1628	37.0639	36.5450
q_4	12.8480	13.0691	13.4849	13.5367	15.0699
q_5	10.0592	10.2133	10.5001	10.5356	11.5650

We compute the optimal strategies corresponding to different λ's and present the results in Table 3. From the table, we could see that the newsvendor's optimal response to different risk preferences varies quite significantly.

Figure 3 displays the values of q_i $(i = 1, \ldots, 5)$ when λ increases from 1 to 5. It can be seen that when loss aversion rises, the newsvendor tends to purchase fewer options with higher reservation prices but lower execution costs (the"firm" options), and more options with medium reservation prices and execution costs. The newsvendor's purchase quantities of options with lower reservation prices and higher execution costs (the "flexible" options) are generally stable. Interestingly enough, the newsvendor's optimal response changes abruptly when loss aversion changes from $\lambda = 1.5$ to 1.6. When $\lambda = 1.5$, the optimal solution falls in subproblem P_1, and P_2 when $\lambda = 1.6$. Recall that P_i corresponds to the case of $Q_k < d_b \leq Q_{k+1}$.

8. CONCLUSION

In this article, we consider a loss-averse newsvendor who is allowed to purchase options from wholesalers before demand is known and must decide how many of those options to execute after demand is observed. We characterize the newsvendor's optimal ordering strategy to maximize his expected utility and propose an efficient algorithm to compute the optimal strategy. As expected, the newsvendor would purchase more than another loss-averse newsvendor who has no supply options and less than a risk-neutral newsvendor who has supply options.

There are many questions to be explored in the future. Since every newsvendor has some sort of capacity constraints, and the cost of dealing with each option supplier can be viewed as a fixed cost, it would be interesting to reformulate the problem using capacitated option contracts with fixed ordering costs. The decision version of this problem is $\mathcal{NP}$ -hard [3], thus one might hope to find efficient heuristics for the problem. Another problem worth considering concerns general risk-averse utility functions. Loss aversion is a special case of risk aversion. The general case would be more intriguing, yet much more difficult to tackle.

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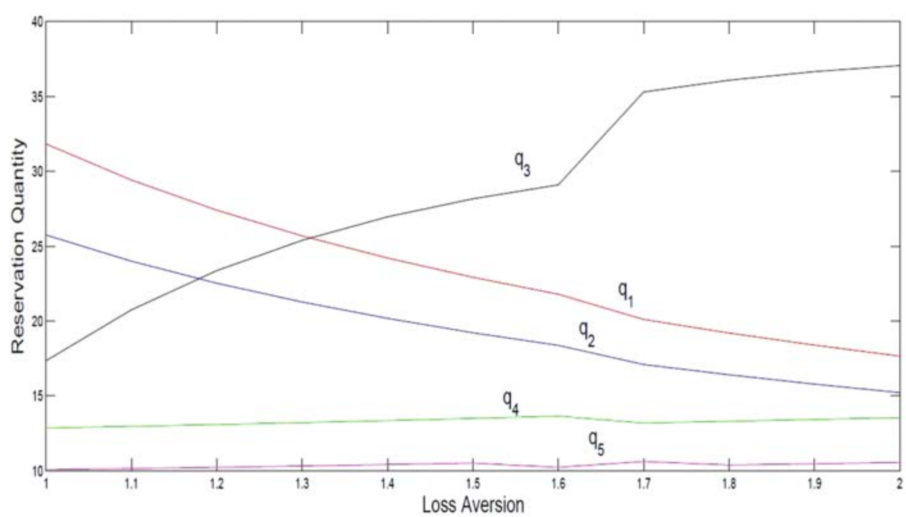

Figure 3. The reservation quantities with different levels of loss aversion. q_1, q_2, q_4, q_5 decrease and q_3 increases with loss aversion. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

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