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This article considers a single-period, multiple-supplier procurement problem with capacity constraints and fixed ordering costs. The buyer can procure from suppliers by signing option contracts with them to meet future uncertain demand. It can purchase from the spot market for prompt delivery at an uncertain price. The objective is to find the optimal portfolio of option contracts with minimal total expected procurement cost. Three cases are discussed. For the case with constant capacity constraints and fixed ordering cost, a dynamic programming approach is used to build a cost function that is strong CK-convex and characterize the structure of the optimal procurement policy, which is similar to the (s, S) policy. However, there is no efficient algorithm for the calculation of the critical parameters or the optimal solution. For the remaining two more restricted cases, one with only capacity constraints (yet zero ordering cost) and the other one with positive ordering cost (yet without capacity constraint), two polynomial algorithms are provided that are able to solve each of them, respectively.

Keywords: Capacitated option contract, procurement, risk management

1. Introduction

Procurement has become an important function for many companies. For example, a computer manufacturer may order key components such as memory chips and hard drives from suppliers. As the cost of the final products is directly affected by the purchasing price of the components, it is critical for the manufacturer to find an effective procurement strategy. Many standard commodity products (electricity, steel, agricultural products, etc.) are available from multiple suppliers as well as the spot market. The spot market, with infinite capacity, provides the buyer with "on the day" purchasing and delivery. However, the spot price is often volatile. Over-reliance on the spot market might incur overwhelming price risk.

One traditional procurement strategy is to sign long-term contracts, also called a fixed commitment contract, with suppliers. This type of contract ensures the buyer a fixed supply quantity in the future at a unit price agreed on by both the supplier and the buyer; however, it may mean a huge inventory risk for the buyer. Besides the fixed commitment contract, another way to reduce inventory and price risks is to use flexible contracts, of which option contracts are a typical example. In option contracts, the buyer is allowed to buy a certain amount of commodities from suppliers at a pre-negotiated price when demand is realized.

The "flexibility" of option contracts lies somewhere between that of a fixed price contract and that of purchasing from the spot market. Making proper use of flexible contracts helps the buyer achieve a better trade-off between inventory risk and price risk, so that the expected procurement cost can be reduced. To better manage the supply, the buyer should sign option contracts simultaneously with several suppliers and uses the spot market as a backup supply source. This purchasing practice has been adopted in industry. For example, Hewlett-Packard has developed a Procurement Risk Management (PRM) program to build contract portfolios with its suppliers (Nagali *et al*., 2008). The company has also applied PRM in the procurement of electricity and memory chips (Billington, 2002).

Martínez-de-Albéniz and Simchi-Levi (2005) examine a multi-period portfolio approach. Taking into account the presence of a spot market, they design effective portfolio contracts and find the optimal replenishment policy that maximizes the buyer's expected profit. They analyze the condition of the optimal procurement decisions but do not provide mechanisms to solve those decisions. Fu *et al*. (2010) propose a mechanism to obtain the optimal procurement solution with the use of an option portfolio under the assumption that the demand and spot price are random and possibly correlated. The models in these

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works are constructed based on the assumption that *none of the option contracts is capacitated and the fixed ordering cost is zero*. However, all suppliers are capacity constrained in one way or another. Also, the buyer pays a fixed ordering cost, which includes costs involved in building and maintaining relationships with suppliers, preparing legal documents, conducting preparatory work, etc., for every option contract it signs with suppliers. Thus, it is important to consider capacity constraints and fixed ordering costs when trying to find the optimal procurement strategy. In practice, it can happen that some companies may have a lower selling price but quite a limited capacity. Furthermore, including the fixed cost may impact the decision of how many contracts are to be signed. These two factors will thus interactively impact the final purchasing decisions. Furthermore, as we will demonstrate later in the article, if we use the algorithm proposed by Fu *et al*. (2010) to solve the problem with fixed cost and capacity constraints, the error can be large.

The main purpose of this article is to find an option contract portfolio that is optimal for the buyer (or manufacturer) such that the expected total procurement cost is minimized. We also show how capacity constraints and fixed ordering costs affect the buyer's decision.

The rest of the article is organized as follows. Section 2 provides a brief literature review. Section 3 sets up the basic model of the PRM problem. Section 4 discusses the general case while Sections 5 and 6 provide efficient algorithms to solve the special case with zero ordering cost and the special case in which each supplier has unlimited capacity, respectively. Section 7 concludes with a summary and discussion.

2. Literature review

Several streams of literature are related to our work. One stream focuses on supply chain coordination through the use of contracts. Readers may refer to Lariviere (1999) or Cachon (2003) for a review. Some scholars specifically study the use of option contracts. Barnes-Schuster *et al*. (2002) illustrate that an option contract provides flexibility for the buyer and develop conditions on the cost parameters such that channel coordination can be achieved. Cheng *et al*. (2008) develop an option model to quantify and price a flexible supply contract.

Serel *et al*. (2001) investigate the rational actions of the buyer and the supplier(s) in the presence of a spot market for two types of periodic review inventory control policies: the two-number policy and the base stock policy. Wu *et al*. (2002) model a negotiable option contract arrangement between one seller and multiple buyers and derive the seller's optimal biding and buyer's optimal contracting strategies in a Stackelberg game with the seller taking the lead. We use a two-part contract price structure similar to the one used in that paper. Kleindorfer and Wu (2003) provide a review of the theory and practice of option contacts in a B2B market. Spinler *et al*. (2003) consider a similar problem and generalize it to the state-dependent case, showing that the model developed by Wu *et al*. (2002) can be extended to the case with stochastic costs and demand. Wu and Kleindorfer (2005) extend the model to multiple sellers and integrate contract procurement markets with spot markets and assume that the demand is dependent on price but is not exogenously generated. Inderfurth and Kelle (2010) compare a combined purchasing policy with single sourcing options and show that using combined sourcing can be advantageous in many cases.

For the case where both demand and spot price are random and exogenous, Golovachkina and Bradley (2003) consider a model consisting of one buyer and one supplier and study whether option contracts and a spot market can coordinate the supply chain. Martínez-de-Albéniz and Simchi-Levi (2005) examine a multi-period portfolio approach. In the companion paper, Martínez-de-Albéniz and Simchi-Levi (2009) consider the competition behavior among suppliers under the single-period framework. Martínez-de-Albéniz and Simchi-Levi (2006) study the trade-off faced by a manufacturer taking into account not only expected profit but also the associated risk and apply a mean-variance analysis to the procurement contracts. Martínez-de-Albéniz (2009) presents two models where the supplier portfolios are optimized to manage demand risk. Haksöz and Seshadri (2007) give a review of the use of the spot market to manage procurement in a supply chain. Fu *et al*. (2010) propose a mechanism to obtain the optimal option procurement solution. They also investigate the benefit of using a portfolio of option contracts instead of sticking to one fixed price contract and find that two carefully selected option suppliers are enough for the buyer to achieve a near-optimal performance in procurement. For the multi-period procurement problem, Fu *et al*. (2012) extend the model of Martínez-de-Albéniz and Simchi-Levi (2005) and allow the dynamic adjustment of contract quantity in each period and obtain the optimality property of replenishment policy. Our article is an extension of Fu *et al*. (2010), which imposes setup cost and capacity constraints.

Another stream of research related to our work focuses on stochastic inventory management with capacity constraints and fixed ordering costs. For the multi-period inventory management problem with fixed ordering costs, a two-parameter (s, S) policy is optimal. The (s, S)-like policy, however, fails to work in the case with capacity constraints (see Chen and Simchi-Levi (2004) and Wijingaard (1972)). Federgruen and Heching (1999) address the multi-period inventory problem with the objective of maximizing the total profit over a finite or infinite horizon and show that the base stock list price policy is optimal. They also point out that the optimal policy has the same structure as the case with capacity constraints. Chen and Simchi-Levi (2004) consider a finite horizon and periodic review model with fixed ordering cost. Gallego and Scheller-Wolf (2000) consider a periodic review inventory problem with fixed ordering cost and capacity constraints. They develop a new concept, known as the strong CK-convex function, and give a rather explicit structure of the optimal ordering policy. We show that their methodology can be applied to our scenario.

While we study the procurement problem from the perspective of the buyer, one may wonder whether the portfolio approach also benefits suppliers. Brown and Lee (1998) and Burnetas and Ritchen (2005) discuss the question briefly and show that the portfolio of an option and a fixed price contract benefits both the buyer and supplier, thus creating a "win–win" situation. Buy-back contracts, similar to option contracts, are beneficial to both buyers and suppliers, as shown in Pasternack (1985).

3. PRM problem: the basic model

Consider a buyer facing uncertain demand for a single product. There are n suppliers in the market. Supplier i offers an option contract with *unit reservation price* c_i and *unit execution price* h_i. Each supplier has limited capacity. Let $Q_i \in (0, \infty)$ denote the capacity of supplier i. Furthermore, a fixed cost is incurred if a supplier is selected and we let $K_i \in [0, \infty)$ denote the fixed ordering cost for option contract i. The buyer can choose to procure from these suppliers by signing certain option contracts or tap into the spot market and purchase the products at a random *spot price* P_s. The demand and spot price are independent and their cumulative distribution functions are available and denoted by $F(D)$ and $G(P_s)$, respectively.

The decision process of option procurement consists of two stages. In the first stage, facing uncertain demand and spot price, the buyer decides to purchase options from n option suppliers by reserving quantity q_i with unit price c_i from supplier i. In the second stage, once the demand and spot price are realized, the buyer determines the execution quantity that cannot exceed the quantity reserved in stage 1, with unit execution price h_i from supplier i. The excess demand is realized using the spot market.

We reindex the suppliers such that $h_1 < h_2 < \cdots < h_n$ and use this index throughout the article. The first supplier offers an option contract with $h_1 = 0$, which is a fixed price (firm commitment) contract. Denote the indicator function by $I\{A\}$, which assumes a value of one if event A is true and a value of zero otherwise. Our problem can be formulated as follows:

(PRM-CK):

$$\min_{q_i:q_i\geq 0, i=1,\ldots n} \left\{ \sum_{i=1}^{n} c_i q_i + E_{D,P_s} \times \left[\min_{q_i',z} \left(\sum_{i=1}^{n} (K_i I\{q_i > 0\} + h_i q_i') + P_s z \right) \right] \right\}$$

$$\text{subject to } 0 \leq q_i' \leq Q_i, \quad \forall i = 1, \ldots, n,$$
$$\sum_i q_i' + z = D, \quad \forall i = 1, \ldots, n,$$
$$q_i \geq 0, \quad \forall i = 1, \ldots, n.$$

Obviously, the problem (PRM-CK) is convex in $q_i, i = 1, \ldots, n$.

For the purpose of minimizing the procurement cost, the buyer needs to make a trade-off between price and flexibility. It can choose the appropriate portfolio of procurement contracts from the option suppliers with either low flexibility (high reservation cost, relatively low execution cost) or high flexibility (low reservation cost, relatively high execution cost). The spot market can also be regarded as a supplier—one that provides an option contract with a fully flexible zero reservation cost but an uncertain and usually highly fluctuating price. The procurement problem is to find the optimal decision in the first stage $q_i, i = 1, \ldots, n$. In this article, an "order" refers only to making a reservation. After the demand D and spot price P_s are realized in the second stage, the optimal strategy for the buyer follows a greedy rule: execute the contracts to satisfy the demand starting from the lowest to highest execution price h_i and realized spot price P_s. We assume that the spot market has infinite capacity; thus the option contracts with a higher execution price than the realized P_s will not be executed. The following two definitions in Fu *et al*. (2010) will also be used in our article.

Definition 1. Contract i is called *active* in a solution of the option procurement problem if $q_i > 0$, where q_i denotes the amount of options reserved from contract i in this solution. For two active contracts i and j with $i < j$, if $q_r = 0$ for all r with $i < r < j$, then contracts i and j are called *consecutive active* contracts. Furthermore, active contract i is called the *last active* contract, if $q_r = 0$ for all with $r > i$.

For better illustration, we introduce another cost parameter.

Definition 2. Let $h_i' = E[\min(h_i, P_s)]$, for $i = 1, \ldots, n$, and $h_{n+1}' = E[P_s]$, then we call h_i' the expected (effective) execution price.

For an option contract reserved in stage 1, if its execution price is higher than the realized spot price P_s, the buyer will not exercise it. Instead, it will purchase from the spot market if needed. We can also interpret the expected execution price in another way. The execution price is implemented as a price "cap," the maximum price that the buyer would pay for execution. If the realized spot price is lower than the execution price, the supplier will lower the execution price to the same level as the spot price to entice the buyer to execute. Thus, the expected execution price of option i is $E[\min(h_i, P_s)]$. Note that the procurement cost to the buyer is the same in both settings. In this article, we take the second settings for ease of interpretation. Whatever the

Table 1. Model extensions and assumptions

	PRM-CK	*PRMC*	*PRMS*
Capacity constraints	Constant capacitated	Capacitated; supplier-dependent	Non-capacitated
Ordering cost	Non-zero and constant	Zero	Non-zero and contract-dependent

realized spot price, we always execute at the expected execution price.

Recall that we have already reindexed the suppliers such that $h_1 < h_2 < \cdots < h_n$. Obviously, this order holds even if h_i is replaced by h'_i. We can also add the spot market as supplier $n+1$ to the sequence. That is, $h'_1 < h'_2 < \cdots < h'_n < h'_{n+1} = E[P_s]$.

We extend the basic PRM model (Fu *et al.*, 2010) by incorporating the capacity constraints of suppliers and the fixed ordering costs of the buyer. Generally, there are three cases.

1. PRM with both capacitated option contracts and fixed ordering costs (PRM-CK).
2. PRM with capacitated option contracts but zero fixed ordering cost (PRMC).
3. PRM with non-zero ordering cost but without capacity constraints (PRMS).

Table 1 shows the model extensions and assumptions.

For Case 1 (PRM-CK), we will provide the structure of the optimal ordering strategy, which is similar to the (s, S) policy. However, there is no efficient algorithm for the calculation of the critical parameters or the optimal solution. For Case 2 (PRMC) and Case 3 (PRMS), efficient algorithms for finding optimal solutions are provided. Moreover, we will allow the capacities (the fixed ordering costs) for each option to vary in the second (third) case.

4. Case 1: PRM-CK

4.1. *The model and assumptions*

In the general option PRM problem, the buyer incurs a fixed setup cost for each order and there may be a finite upper bound on the reservation quantity that option suppliers are able to offer. The spot market, however, has infinite capacity. We assume that the fixed ordering cost, denoted as K, is identical for all option contracts and is incurred at reservation instead of execution. We also assume that the capacity constraint is identical for all option suppliers, denoted as C (i.e., $Q_i = C$ for $i = 1, \ldots, n$). Reservation is the compulsory part of an option contract, while execution is optional and involves buying and delivering only. Thus, a variable execution fee is enough to describe the cost incurred for execution. We assume that there is no fixed cost for purchasing from the spot market.

To obtain the optimal solution of the PRM-CK problem, we consider the reservation quantity of each option supplier one by one, from $i = 1$ to n. We adopt the methodology used by Gallego and Scheller-Wolf (2000), which gives the optimal ordering policy for the multi-period capacitated inventory problem with stationary stochastic demand and fixed ordering cost. Our single period, multi-supplier procurement problem with capacity constraints and fixed ordering cost can be formulated in an appropriate way such that the cost functions satisfy a similar property—strong CK-convexity. The optimal ordering policy can be applied to our model. This policy allows us to determine the optimal procurement quantity of each option supplier.

4.2. *Dynamic programming*

Define

x_i: total reservation quantity of option contracts $1, \ldots, i-1$;

y_i: total reservation quantity of option contracts $1, \ldots, i$. $y_i = x_{i+1}$;

$y_i - x_i\ (= q_i)$: the reservation quantity of option contract i, a decision variable;

$f_i(x)$: the minimum expected procurement cost of option contracts $i, i+1, \ldots, n$, with current reserved level x and facing uncertain demand and spot price.

We seek the form of $f_1(0)$, which returns the procurement policy of the (PRM-CK) problem.

$$\begin{aligned} cost_i(y_i) &= (c_i + h'_i)y_i, \\ L_i(y_i) &= h'_i E(y_i - D)^+, \\ f_i(x_i) &= -cost_i(x_i) + L_i(x_i) \\ &\quad + \inf_{y \in [x_i, x_i + C]}\{K \times I\{y > x_i\} + G_i(y)\}, \quad (1) \\ G_i(y_i) &= cost_i(y_i) - L_i(y_i) + f_{i+1}(y_i), \quad (2) \end{aligned}$$

for $i = 1, \ldots n$; and

$$f_{n+1}(x_{n+1}) = E(P_s) \times E(D - x_{n+1})^+$$

Note that:

$$\begin{aligned} cost_i(y_i) - L_i(y_i) &= (c_i + h'_i)y_i - h'_i E(y_i - D)^+, \\ &= c_i \times y_i + h'_i \times E[\min(D, y_i)]. \end{aligned}$$

Please note that since $G_i(y_i)$ denotes the expected procurement cost from option contract i if the reservation quantity equals to y_i, and y_i is the total reservation quantity of option contracts 1 to i, to calculate the value of $f_i(x_i)$ we have to cancel out the procurement cost of option contracts 1 to $i-1$; i.e., the term $[cost_i(x_i) - L_i(x_i)]$ from $G_i(y_i)$.

4.3. *Strong CK-convexity*

Definition 3. (Gallego and Scheller-Wolf, 2000): Given a non-negative constant K, we refer to the function $G:\mathbb{R}\to\mathbb{R}$ as a strong CK-convex function if for any $y, 0\le a<\infty, 0<b<\infty, 0\le z\le C$, we have

$$K+G(y+z)\ge G(y)+\frac{z}{b}\{G(y-a)-G(y-a-b)\}.$$

If $0\le z\le\infty$, then the function is a strongly K-convex. If both $a=0$ and $0\le z\le\infty$, then it is a well-known K-convex function defined by Scarf (1960). It is easy to see that any convex function is also a strongly K-convex function, and any strongly K-convex function is also a strongly CK-convex function.

By reformulation of the definition:

$$\frac{K+G(y+z)-G(y)}{z}\ge\frac{G(y-a)-G(y-a-b)}{b},$$

it is easy to think of strong CK-convexity by the following geometrical interpretation (see Fig. 1): The slope of the line linking $(y-a-b, G(y-a-b))$ with $(y-a, G(y-a))$ is always less than or equal to the slope of the line linking $(y, G(y))$ with $(y+z, G(y+z)+K)$ for any $0\le z\le C$.

Proposition 1. Gallego and Scheller-Wolf (2000). *If G_1 is strongly CK-convex, and G_2 is convex, then G_1+G_2 is strongly CK-convex.*

Lemma 1. *$L_i(y)=h'_iE(y-D)^+$ is convex for every $i=1,\ldots,n$.*

Proof.

$$\begin{aligned}L_i(y)&=h'_iE(y-D)^+\\&=h'_i\times\left(\int_{-\infty}^{y}yf(D)\mathrm{d}D-\int_{-\infty}^{y}Df(D)\mathrm{d}D\right),\end{aligned}$$

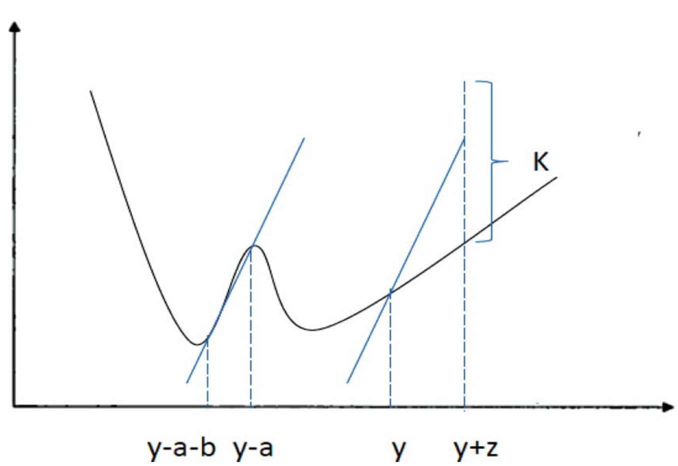

Fig. 1. Strongly K-convex function (color figure provided online).

$$\begin{aligned}\frac{\mathrm{d}L_i(y)}{\mathrm{d}y}&=h'_i\left(\int_{-\infty}^{y}f(D)\mathrm{d}D+yf(y)-yf(y)\right)\\&=h'_i\times\int_{-\infty}^{y}f(D)\mathrm{d}D,\\\frac{\mathrm{d}^2L_i(y)}{\mathrm{d}y^2}&=h'_i\times f(y)\ge 0.\end{aligned}$$

■

Lemma 2. Gallego and Scheller-Wolf (2000). *Given non-negative C and K, and a strong CK-convex function $G:\mathbb{R}\to\mathbb{R}$, define*:

$$H(x)\overset{\text{def}}{=}\inf_{y\in[x,x+C]}\{K\times I\{y>x\}+G(y)\}.$$

Then $H(x)$ is also strongly CK-convex.

Please note that $H(x)$ represents the optimal cost function after the new order is finished, given that the current inventory level is x.

Theorem 1. *In Equations* (1) *and* (2), *$G_i(\cdot)$ and $f_i(\cdot)$ are strongly CK-convex for all $i=1,\ldots,n$, where K and C denote the fixed setup cost and capacity of each option contract.*

Proof. The proof is based on mathematical induction.

First, we consider $f_{n+1}(x)$ and $G_n(y)$. Similar to Lemma 1, we can show that:

$$f_{n+1}(x)=E(P_s)\times E(D-x)^+,$$

is convex in x. $G_n(y)=cost_n(y)-L_n(y)+f_{n+1}(y)$ is also convex in y, because:

$$\frac{\mathrm{d}^2G_n(y)}{\mathrm{d}y^2}=(E(P_s)-h'_n)\times f(y)\ge 0.$$

Second, we consider $f_n(x)$ and $G_{n-1}(y)$. We know that if $G_n(y)$ is convex, it is also CK-convex. From Lemma 2:

$$\inf_{y\in[x,x+C]}\{K\times I\{y>x\}+G_n(y)\},$$

is strongly CK-convex in x. So

$$\begin{aligned}f_n(x)=&-cost_n(x)+L_n(x)\\&+\inf_{y\in[x,x+C]}\{K\times I\{y>x\}+G_n(y)\},\end{aligned}$$

is also strongly CK-convex in x because the first two terms are both convex.

Then

$$\begin{aligned}G_{n-1}(x)&=cost_{n-1}(x)-L_{n-1}(x)+f_n(x)\\&=(c_{n-1}+h'_{n-1}-c_n-h'_n)x+(h'_n-h'_{n-1})E(x-D)^+\\&\quad+\inf_{y\in[x,x+C]}\{K\times I\{y>x\}+G_n(y)\},\end{aligned}$$

is also strongly CK-convex in x again because the first two terms are both convex.

Then by backward mathematical induction, we can show that $G_i(y)$ and $f_i(x)$ are strongly CK-convex for $i=1,2,\ldots,n$. ■

4.4. *The decision process and the optimal policy*

Rewrite the equation in dynamic programming as

$$f_i(x_i) = -cost_i(x_i) + L_i(x_i) + \inf_{y\in[x_i,x_i+C]}\{K \times I\{y > x_i\} + G_i(y)\},$$
$$G_i(y_i) = cost_i(y_i) - L_i(y_i) + f_{i+1}(y_i).$$

Given x_i, y_i is determined by

$$y_i = \operatorname{argmin}_{y\in[x_i,x_i+C]}\{K \times I\{y > x_i\} + G_i(y)\}. \quad (3)$$

We start from $x_1 = 0$ and sequentially determine all y_i, $i = 1, \ldots, n$. Note that $x_{i+1} = y_i$, and $y_i - x_i$ denotes the quantity to be ordered from option supplier i. The challenge is to find y_i from Equation (3) for a given x_i.

Define:

$$G_i^* \stackrel{\text{def}}{=} \inf_{y\geq 0} G_i(y)$$
$$S_i \stackrel{\text{def}}{=} \{y \geq 0 | G_i(y) = G_i^*\}$$
$$\tilde{G}_i(x) \stackrel{\text{def}}{=} K + \inf_{\{x\leq y\leq x+Q_i\}} G_i(y)$$
$$A_i(x) \stackrel{\text{def}}{=} \tilde{G}_i(x) - G_i(x)$$
$$s_i \stackrel{\text{def}}{=} \inf\{x | A_i(x) \geq 0\}$$
$$s_i' \stackrel{\text{def}}{=} \max\{x \leq S_i | A_i(x) \leq 0\}$$

By the definition of s_i' we have $0 \leq s_i' \leq S_i$. Note that $A_i(S_i) = \tilde{G}_i(S_i) - G_i(S_i)$ and for any $y \geq 0$ we have $G_i(S_i) \leq G_i(y)$ so it follows that $A_i(S_i) \geq 0$ and we can come up with $0 \leq s_i \leq S_i$. Also, by definition $A(x_i) < 0$ on $x_i < s_i$ and $A(x_i) > 0$ on $s_i' < x_i \leq S_i$, and thus we have $0 \leq s_i \leq s_i' \leq S_i$. Furthermore, we claim that S_i is finite for $i = 1, \ldots n$. This is due to the fact that the slope of $G_i(y)$ approaches c_i as $y \to \infty$, so $\lim_{y\to\infty} G_i(y) = \infty$ for $i = 1, \ldots n$.

In order to illustrate the optimal procurement policy, we introduce the following results from Gallego and Scheller-Wolf (2000).

Lemma 3.

1. *$G_i(x)$ is non-increasing on $(-\infty, s_i']$ and strictly decreasing on $(-\infty, s_i)$.*
2. *$A_i(x) \geq 0 \forall x > s_i'$.*

Define:

$$I_i^+ \stackrel{\text{def}}{=} I\{s_i' - C > s_i\}$$
$$I_i^- \stackrel{\text{def}}{=} I\{s_i' - C < s_i\}$$
$$G_i^C(x) \stackrel{\text{def}}{=} K + G_i(x + C)$$
$$\bar{G}_i(x) \stackrel{\text{def}}{=} K + \inf_{\{s_i'\leq y\leq x+C\}} G_i(y), \quad s_i' - C \leq x \leq s_i'$$

Theorem 2. *Given non-negative C and K, and $G_i : \mathbb{R} \to \mathbb{R}$ is strongly CK-convex, let*

$$H_i(x) \stackrel{\text{def}}{=} \inf_{y\in[x,x+C]}\{K \times I\{y > x\} + G_i(y)\} = \min\{G_i(x), \bar{G}_i(x)\}.$$

Then

$$H_i(x) = \begin{cases} G_i^C(x), & x < \min\{s_i, s_i' - C\} \\ \{\bar{G}_i(x)\}I_i^- + \min\{G_i(x), G_i^C(x)\}I_i^+, & \min\{s_i' - C, s_i\} \leq x < \max\{s_i' - C, s_i\} \\ \min\{G_i(x), \bar{G}_i(x)\}, & \max\{s_i' - C, s_i\} \leq x \leq s_i' \\ G_i(x), & s_i' < x \end{cases} \quad (4)$$

In the work of Gallego and Scheller-Wolf (2000), $H(x)$ describes an explicit structure of the optimal ordering policy for a finite planning horizon problem. The parameter space is divided into four regions. By examining the inventory level at the start of each period, the buyer optimally decides to order the full capacity or nothing or bring the inventory to a specified level. In our model, we transform the ordering decision made in each "period" into the decision of how much to reserve from each "supplier" (i.e., the reservation quantity). The technical analysis is similar to that in Gallego and Scheller-Wolf (2000). We thus do not provide detailed analysis/explanation.

Specifically, for option supplier i.

1. If $x_i < \min\{s_i, s_i' - C\}$, $H_i(x) = K + G_i(x + C)$ and so it is optimal to order from option supplier i and use up its capacity.
2. If $\min\{s_i, s_i' - C\} \leq x_i \leq s_i'$, it is optimal to order nothing or to bring the total ordering quantity up to a specified level, which is decided by the values of $G_i(x_i)$, $G_i^C(x_i)$, and $\bar{G}_i(x_i)$, as indicated by Equation (4).
3. If $x_i > s_i'$, $H_i(x) = G_i(x)$ and so it is optimal not to order from option i.

Starting with inventory level $x_1 = 0$, we determine the quantity to be ordered from option supplier 1 following the optimal policy discussed above, which fixes y_1; given $x_2 = y_1$, we similarly determine the ordering quantity from option supplier 2. Repeating this procedure for all options suppliers will give us the optimal procuring quantity.

There is no easy way to calculate s_i and s_i' even though they are critical for the optimal ordering policy. Conducting a numerical study is one option.

5. Case 2: PRMC problem

5.1. *The model and assumptions*

This section considers the PRMC problem. It is a special case of PRM-CK when $K = 0$ though the capacity in each option is allowed to vary. We show that an efficient algorithm can be provided to solve the problem optimally.

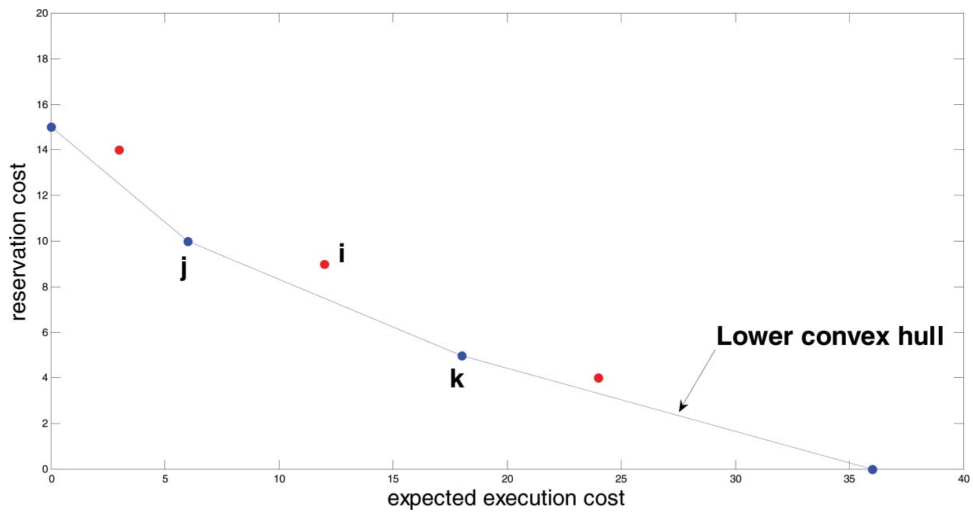

Fig. 2. Dominated and dominating option contracts (color figure provided online).

We represent all of the option contracts by their cost parameters in a two-dimensional plane, as shown in Fig. 2. Denote the set of option contracts by points $(h_i', c_i)'$s, $i = 1, \ldots, n$, and the spot market by point $(h_{n+1}', 0)$.

Definition 4. An option i is *dominated* if the following situations occur:

1. There is an option k such that $h_i' < h_k'$ and $c_i + h_i' > c_k + h_k'$; i.e., point i lies to the left of point k and above the straight line passing through point k with slope -1.
2. Or there are options j and k such that $h_j' < h_i' < h_k'$ and
$$\frac{h_i' - h_j'}{h_k' - h_j'} c_j + \frac{h_k' - h_i'}{h_k' - h_j'} c_k < c_i;$$
i.e., point i lies between points j and k and above the straight line passing through them.

Option j and k are called *dominating* options of option i.

As can be seen from Fig. 2, an option contract is dominated if and only if it lies above the lower convex hull of the option contract set, and the lower convex hull only refers to the portion with downward slope between zeroand one.

For the case where all option suppliers have unlimited capacities, Fu *et al*. (2010) show that the active option contracts in the optimal solution lie on the lower convex hull; i.e., they are not dominated. Hence, their algorithm is efficient and could compute the optimal procurement solution in polynomial time. However, the algorithm fails when it is applied to the PRMC, problem as we will show later in Example 1 (Section 5.4), since $q_i \le Q_i$ does not always hold in the optimal solution to the PRM problem.

Thus, before considering the PRMC problem, we first consider an easier problem. Let us call it the PRMC(S,I) problem. In the PRMC(S,I) problem, there is a subset of suppliers, S, and a positive integer, I, such that for each supplier i in S, we already have $q_i = Q_i$ and we have to try and select more suppliers from those suppliers not in S (including spot market) with total amount I and assume that the capacities of these suppliers are unlimited. Our purpose is to satisfy the demand with minimal total cost. More specifically, suppose that there are n suppliers $1, 2, \ldots, n$. We define (h_i', c_i) similarly for $i = 1, \ldots, n+1$. There is no capacity constraint for those contracts not in S. Given a positive integer I and a subset $S \subseteq \{1, 2, \ldots, n\}$, the goal of the PRMC($S$,$I$) problem is to find a procurement strategy $q = (q_1, q_2, \ldots, q_{n+1})$ such that the following hold.

1. For every $i \in S$, we have $q_i = Q_i$.
2. $\sum_{i=1, i \notin S}^{n+1} q_i = I$.
3. Let $C(q)$ denote the expected procurement cost when the procurement strategy is q. For any other strategy q' such that conditions 1 and 2 are satisfied, we have
$$C(q) \le C(q')$$
Here the cost function $C : (Z^+)^{n+1} \to R$ is defined as
$$C(q) = \sum_{i=1}^{n+1} \sum_{j=1}^{q_i} \left(c_i + h_i' \Pr\left(D \ge \sum_{k=1}^{i-1} q_k + j \right) \right) + \sum_{i=1}^{\infty} h_{n+1}' \Pr\left(D \ge \sum_{j=1}^{n+1} q_j + i \right).$$
The first part, $\sum_{i=1}^{n+1} \sum_{j=1}^{q_i} (c_i + h_i' \Pr(D \ge \sum_{k=1}^{i-1} q_k + j))$, corresponds to the cost related to the options we reserved, and the second part, $\sum_{i=1}^{\infty} h_{n+1}' \Pr(D \ge \sum_{j=1}^{n+1} q_j + i)$, corresponds to the cost of buying additional items from the spot market when the demand is larger than the quantity we reserved. Note that the spot market is also viewed as a supplier whose $c_{n+1} = 0$; thus,

the cost function can be equivalently written as

$$C(q) = \sum_{i=1}^{n} \sum_{j=1}^{q_i} \left(c_i + h_i' \Pr\left(D \geq \sum_{k=1}^{i-1} q_k + j \right)\right) + \sum_{i=1}^{\infty} h_{n+1}' \Pr\left(D \geq \sum_{j=1}^{n} q_j + i \right),$$

which does not depend on the value of q_{n+1}. In this way, we neglect q_{n+1} and write $C(q_1, \ldots, q_n)$ when it is obvious from the context.

Definition 5. Contract i is called *saturated* in a procurement solution if $q_i \geq Q_i$.

5.2. *The PRMC(*S,I*) problem and its optimal solution*

We propose the following algorithm to solve the PRMC(S,I) problem. Since the algorithm reserves the units one by one, we call the algorithm the Walk-and-Stop algorithm.

The Walk-and-Stop Algorithm

Step 1: Set $i = 0$ and $q = 0$. For $k = 1, \ldots, n + 1$, we set $q_k = Q_k$ if $k \in S$, and $q_k = 0$ otherwise.
Step 2: Find $j \geq i$ and j not contained in S that minimizes:

$$C(q_1, \ldots, q_j + 1, \ldots, q_{n+1}).$$

Step 3: Set $q = q + 1$, $q_j = q_j + 1$ and $i = j$. If $q < I$ go to Step 2; otherwise, stop and output $q = (q_1, \ldots, q_{n+1})$.

Theorem 3. *The Walk-and-Stop algorithm outputs an optimal solution to the PRMC(S,I) problem.*

Proof. See the Appendix. ■

Please note that the PRMC problem is much harder than the PRMC(S,I) problem due to the capacity constraints. The following theorem is used to cope with this problem.

Theorem 4. (The Saturating Theorem.) *Given a subset S of* $\{1, \ldots, n\}$, *suppose that all contracts in S are saturated in the optimal solution to the PRMC problem. Run the Walk-and-Stop algorithm for the PRMC(S,I) problem. If we find that contract i is the first contract that is not in S and* $q_i \geq Q_i$ *then contract i should be saturated in the optimal solution to the PRMC problem.*

Proof. See the Appendix. ■

The Saturating Theorem makes it possible for us to compute the optimal solution to the PRMC problem: we begin with $S = \emptyset$ and run the Walk-and-Stop algorithm to solve the PRMC(S,I) problem with I being a very large number. If there is no saturated supplier, we have reached the optimal solution. Otherwise, we can run the algorithm again and gradually insert elements into S. In this way, we can obtain the set S consisting of all saturated contracts in the optimal solution of the PRMC problem.

5.3. *An optimal algorithm for the PRMC problem*

We present an algorithm for the PRMC problem based on the Saturating Theorem. This algorithm proceeds by identifying the saturated contracts one by one, and after the set of saturated contracts is updated a new run is performed to determine the corresponding reservation strategy.

Algorithm for the PRMC problem

Step 1: Run the Walk-and-Stop algorithm for PRMC(Ø,I) with $S = \emptyset$ and I being a large number. If the algorithm ends up without any supplier i with $q_i > Q_i$ stop.
Otherwise, let the first saturated contract be i, let $S = \{i\}$, and go to Step 2.
Step 2: Run the Walk-and-Stop Algorithm for PRMC(S,I) where I is a large number. If the algorithm stops with no new saturated supplier, stop. Otherwise, let the first new saturated supplier contract be i, let $S = S \cup \{i\}$ and continue Step 2.

Now let us analyze the algorithm. First, observe that when the algorithm stops, all contracts in S must be saturated based on the Saturating Theorem. We increase $|S|$ by one each time we come back to Step 2. It takes at most n runs of the algorithm for the PRMC(S,I) problem to find all of the new saturated suppliers. Hence, the complexity depends on the efficiency in solving each PRMC(S,I) problem. Currently, the algorithm for the PRMC(S,I) problem is pseudo-polynomial as it depends on the value of I. In the next subsection, we will provide a polynomial algorithm to solve the problem.

Different from previous literature on PRM that use dynamic programming as the main approach, we employ a different approach that moves back and forth. This is less efficient, but dynamic programming might not be applied here since the existences of capacity constraints remove the desired property on which dynamic programming rests.

5.4. *Time complexity*

In this subsection, we focus on the time complexity of the algorithm for the PRMC(S,I) problem. First, in Step 2, we should find among $j \geq i$ and $j \notin S$ the j that minimizes:

$$C(q_1, \ldots, q_j + 1, \ldots, q_{n+1}).$$

This is done by comparing $C(q_1, \ldots, q_u + 1, \ldots, q_n)$ with $C(q_1, \ldots, q_v + 1, \ldots, q_n)$ for any $1 \leq u < v \leq n + 1$ in the following way: Let $A = \sum_{k=1}^{u} q_k$ and

$$\begin{aligned} & C(q_1, \ldots, q_u + 1, q_{u+1}, \ldots, q_n) \\ & \quad - C(q_1, \ldots, q_u, q_{u+1} + 1, \ldots, q_n) \\ & = c_u - c_{u+1} + (h'_u - h'_{u+1}) \Pr(D > A) \\ & = B_u \\ & \ldots\ldots\ldots \\ & C(q_1, \ldots, q_{v-1} + 1, q_v, \ldots, q_n) \\ & \quad - C(q_1, \ldots, q_{v-1}, q_v + 1, \ldots, q_n) \\ & = c_{v-1} - c_v + (h'_{v-1} - h'_v) \Pr(D > A + q_{u+1} + \cdots + q_{v-1}) \\ & = B_{v-1}. \end{aligned}$$

We need to check if $\sum_{k=u}^{v-1} B_k > 0$. This can be done in O(n), assuming that we can find $\Pr(D > x)$ in constant time. In this way, we can find j by repeating the comparison $O(n)$ times. After finding j, for every $k > j$, we can try to find a unique x_j^k such that:

$$\begin{aligned} & C\big(q_1, \ldots, q_j + x_j^k, \ldots, q_n\big) \\ & \quad \le C\big(q_1, \ldots, q_j + x_j^k - 1, \ldots, q_k + 1, \ldots, q_n\big), \end{aligned} \tag{5}$$

and

$$\begin{aligned} & C\big(q_1, \ldots, q_j + x_j^k + 1, \ldots, q_n\big) \\ & \quad > C\big(q_1, \ldots, q_j + x_j^k, \ldots, q_k + 1, \ldots, q_n\big). \end{aligned} \tag{6}$$

This is done by performing a binary search of x_j^k over $(0,\ D_{\max}]$, which takes $O(\log D_{\max})$ comparisons, where each comparison costs $O(n)$ time. Here $D_{\max}$ is the maximum possible value of demand. Although $D_{\max}$ could become ∞ theoretically, the demand is always limited in practice, and $\log D_{\max}$ is rather small for reasonable values of maximum demand.

Now we can update the Walk-and-Stop algorithm to the following:

The modified Walk-and-Stop algorithm

Input $S = \emptyset$ and $I = D_{\max}$

Step 1: Set $i = 1$, $q_k = Q_k$ for every $k \in S$, and $q_k = 0$ otherwise for every $1 \le k \le n$.

Step 2: Among $j \ge i$ and j not in S, find the j that minimizes:

$$C(q_1, \ldots, q_j + 1, \ldots, q_{n+1}).$$

For every $k > j$ and k not in S find x_j^k, which was defined earlier in Equations (5) and (6).
Let $x = \min\{x_j^k\}$.

- If $j = n + 1$, stop and output $(q_1, \ldots, q_n)$.
- If $q_j + x \ge Q_j - 1$, set $S = S \cup \{j\}$ and go to Step 1.
- Otherwise, set $q_j = q_j + x$, $i = j$ and go to Step 2.

Table 2. Parameters

	Supplier				
	1	*2*	*3*	*4*	*Spot market*
c_i	10	6.2	2.7	0.9	—
h'_i	6	10	15	24	42
Q_i	6	5	2	4	

Remark 1. The algorithm for the PRMC problem terminates in time $O(n^3 \log D_{\max})$ if we replace the Walk-and-Stop algorithm by the modified Walk-and-Stop algorithm in both steps. Hence, it is a polynomial algorithm. Please see Garey and Johnson (1979, p. 93) for the reference.

5.5. *An Example*

In this subsection, we show how the algorithm we provided above performs with a simple example.

Example 1: Suppose that the demand follows the normal distribution $D \sim N(10, 4)$ and $E[P_s] = 42$. Consider a four-supplier problem with the parameters listed in Table 2.

We start to run the Walk-and-Stop algorithm with $S =$ empty set and find that supplier 1 is the first saturated. We then fix $q_1 = Q_1$ (i.e., $S = \{1\}$) and run the Walk-and-Stop algorithm again and have $q_3 \ge Q_3$. Now we fix $q_3 = Q_3$, set $S = \{1, 3\}$, and run the Walk-and-Stop algorithm for a third time, which yields the results listed in Table 3.

Here for every supplier, we have $q_i \le Q_i$, so the optimal reserving strategy for the PRMC problem is $q = (6, 3, 2, 2)$.

Remark 2. For comparison purposes, we also obtain the optimal reservation strategy for the case without capacity constraints. The optimal reservation strategy corresponding to this case is listed in Table 4.

Clearly, the algorithm of Fu *et al.* (2010) cannot give us the optimal solution for the case with capacity constraint. Furthermore, there is no one-way forward algorithm for solving the capacity problem, as supplier 3 is saturated, so we need to have some strategy to be able to increase q_2 from two to three.

Example 2: In Example 1, we have shown that the capacity constraint will make the optimal solution different from that without capacity constraint. In this example, we demonstrate how the optimal solutions change as we vary

Table 3. Results

	Supplier				
	1	*2*	*3*	*4*	*Spot market*
q_i	6	3	2	2	

Table 4. The optimal reservation strategy

	Supplier				
	1	*2*	*3*	*4*	*Spot market*
q_i	6	2	3	2	

capacity. We choose 10 options (and the spot market), whose reservation costs and execution costs are as listed in Table 5.

We assume that the demand follows a normal distribution $D \sim N(1000, 500^2)$. In addition, we assume that the capacity constraints on different options are identical, except the spot market whose capacity is infinite. Then we compute the optimal solutions $q = (q_1, \ldots, q_n)$ for different constraints Q. The results are shown in Table 6.

6. Case 3: ($Q_i = \infty$ for all i) for the PRMS problem

6.1. *The model and assumptions*

In the option procurement management problem, the buyer incurs fixed ordering costs for using option contracts but the option suppliers do not have any capacity constraint. Here, we provide a polynomial algorithm to solve the problem.

With the presence of fixed ordering cost, the buyer needs to determine whether or not to use a certain option contract. The option contract with high setup cost may not be included in the optimal portfolio, even if its reservation and execution cost are low. Hence, as we will show in Example 3 later in this section, the active option contracts in the optimal solution may not lie on the lower convex hull; i.e., some can be dominated. Nevertheless, we show that those active option contracts in the optimal solution still form a convex hull. Then we use an algorithm to search for the optimal solution.

Theorem 5. *The optimal solution of PRMS must satisfy:*

$$(c_{i_k} - c_{i_{k+1}}) = \left(h'_{i_{k+1}} - h'_{i_k}\right) P\left(D > \sum_{r=1}^{k} q^*_{i_r}\right), \tag{7}$$

for each $k \in \{1, \ldots, |A| - 1\}$, *and*

$$c_{i_{|A|}} = \left(E(P_s) - h'_{i_{|A|}}\right) P\left(D > \sum_{r=1}^{|A|} q^*_{i_r}\right), \tag{8}$$

where $A = \{i_1 < i_2 < \cdots < i_{|A|}\}$ *denotes all of the active contracts, indexed in an increasing order of execution cost, and* $q^*_{i_k}$ *denotes the reservation quantity of option contract* i_k, $k = 1, \ldots, |A|$.

Proof. See the Appendix. ■

Please note that the same optimality condition in Equations (7) and (8) is also valid for the problem with no fixed cost (Fu *et al*., 2010). Furthermore, Theorem 5 implies that if we define:

$$\alpha_{i_k} = \frac{(c_{i_k} - c_{i_{k+1}})}{\left(h'_{i_{k+1}} - h'_{i_k}\right)},$$

then $\alpha_{i_k} \in [0, 1]$ and $\{\alpha_{i_k}\}$ is a non-increasing series. Namely, those active contracts in the optimal solution form a convex hull itself in the two-dimensional space $\{c_i, h'_i\}$, though it may not be a lower envelop convex hull for the two-dimensional space that includes "all contracts."

Theorem 6. *Let* q^* *be a solution that satisfies Conditions* (7) *and* (8) *and denote its active contract set by* $A = \{i_1 < i_2 < \cdots < i_{|A|}\}$. *For ease of exposition, let* $N = |A|$ *and* $(q^*_1, q^*_2, \ldots, q^*_N)$ *denote the reservation quantity of active option contracts. The total procurement cost generated by the solution* q^* *is*

$$\begin{aligned} C_A(q^*) = K_1 + \sum_{i=1}^{N-1} \Bigg(K_{i+1} + h'_i \int_0^{\sum_{r=1}^{i} q^*_r} Df(D)\mathrm{d}D \\ - h'_{i+1} \int_0^{\sum_{r=1}^{i} q^*_r} Df(D)\mathrm{d}D \Bigg) \\ + h'_N \int_0^{\sum_{r=1}^{N} q^*_r} Df(D)\mathrm{d}D \\ + E(P_s) \int_{\sum_{r=1}^{N} q^*_r}^{\infty} Df(D)\mathrm{d}D \end{aligned} \tag{9}$$

Remark 3. There are three groups of terms in Equation (9):

K_1: The immediate cost incurred if we decide to use the first option; that is, the setup cost associate with the first supplier.

$K_{i+1} + h'_i \int_0^{\sum_{r=1}^{i} q^*_r} Df(D)\mathrm{d}D - h'_{i+1} \int_0^{\sum_{r=1}^{i} q^*_r} Df(D)\mathrm{d}D$: The additional cost incurred if we decide to add one more option contract $i + 1$ to the active contract portfolio.

Table 5. Reservation and execution costs

	Options										
	1	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*Spot*
Reservation price c_i	10	9	8	7.1	6	5.1	4	3.1	2	1.1	0
Execution price h'_i	4.5	5.6	6.8	8.1	9.5	11	12.6	14.3	16.1	18.2	20

Table 6. The optimal solutions to the procurement risk management problem with different capacity constraints the bold numbers denote the options that are saturated

Q	*infinity*	*300*	*250*	*200*	*150*	*130*	*115*	*100*
q_1	332	**300**	**250**	**200**	**150**	**130**	**115**	**100**
q_2	184	216	**250**	**200**	**150**	**130**	**115**	**100**
q_3	161	161	177	**200**	**150**	**130**	**115**	**100**
q_4	0	0	0	11	**150**	**130**	**115**	**100**
q_5	136	136	177	**200**	**150**	**130**	**115**	**100**
q_6	0	0	0	0	8	122	**115**	**100**
q_7	96	96	96	98	**150**	**130**	**115**	**100**
q_8	0	0	0	0	0	0	60	**100**
q_9	74	74	74	74	75	81	**115**	**100**
q_{10}	0	0	0	0	0	0	0	0

$h'_N \int_0^{\sum_{r=1}^N q_r^*} Df(D)\mathrm{d}D + E(P_\mathrm{s}) \int_{\sum_{r=1}^N q_r^*}^{\infty} Df(D)\mathrm{d}D$: The procurement cost associated with the last active option and the spot market.

Remark 4. We let $|A|$ be zero when we do not use any option contract. In such a case, the expected procurement cost is $E(P_\mathrm{s}) \times \int_0^\infty Df(D)\mathrm{d}D$.

Since the optimal solution may not be the lower convex hull of the $\{c_i, h'_i\}$ space we cannot apply directly the algorithm in Fu *et al*. (2010) to solve the problem. Nevertheless, through Theorems 6 and Remarks 3 and 4, we are able to provide a polynomial-time algorithm to solve the problem. Note that in the second term of the cost function, there are $N-1$ groups, each one involves $\sum_{r=1}^{i} q_r^*$ for $i = 1, \ldots, N-1$. We shall use the Shortest Monotone Path Algorithm in Section 6.3 to solve the problem. Note also that this type of algorithm was used in Fu *et al*. (2010) to solve a similar problem without setup cost and capacity constraint yet demand and spot price were correlated. Here, we use similar notations as those used in Fu *et al*. (2010).

6.2. *The optimal solution*

First, we generate a directed graph G with n nodes denoting n option contracts and with directed arcs (i, j) joining two nodes i and j, for all $1 \le i < j \le n$. For each arc(i, j), there is a corresponding pair of parameters (k_{ij}, d_{ij}), where k_{ij} is a scalar and d_{ij} is the length of the arc. We use k_{ij} to keep track of the aggregated quantity purchased up until contract i, if i, j are two consecutive active contracts in a solution that satisfy Equations (7) and (8). Namely, we set k_{ij} to be the solution of

$$(c_i - c_j) = P(D > k_{i,j})(h'_j - h'_i),$$

and set it to zero if the equality has no solution. We also use d_{ij} to keep track of a component in the procurement cost function; that is,

$$d_{i,j} = K_j + h'_i \int_0^{k_{i,j}} Df(D)\mathrm{d}D - h'_j \int_0^{k_{i,j}} Df(D)\mathrm{d}D,$$

and set it to zero if $k_{i,j}$ is zero.

An origin node O and an end node E are then added to graph G. Also, we add an arc connecting O to each node in G and an arc that connects each node in G to E. For each arc(O, j), set $k_{O,j} = 0$, and $d_{O,j} = K_j$; for each arc(j, E), set $k_{j,E}$ to be the solution of

$$c_j = P(D > k_{j,E})(E(P_\mathrm{s}) - h'_j),$$

and

$$d_{j,E} = h'_j \int_0^{k_{j,E}} Df(D)\mathrm{d}D + E(P_\mathrm{s}) \int_{k_{j,E}}^{\infty} Df(D)\mathrm{d}D.$$

For the case where a solution may contain no option contract, we add an arc connecting the origin node O to the end node E, with length $d_{O,E} = E(P_\mathrm{s}) \times E(D)$.

A path from node O to node E with k_{ij} being monotonously increasing along the path is called a monotone path. The monotone paths set of graph G and the solutions set defined by Conditions (7) and (8) have a one-to-one mapping. The shortest monotone path corresponds to a procurement solution with the lowest procurement cost. Thus, the PRMS problem can be solved by finding the shortest monotone path from the origin O to the destination E in graph G. The nodes lying on the shortest monotone path are the active contracts in the optimal solution and the corresponding distance equals the optimal procurement cost.

6.3. *The shortest monotone path algorithm*

The algorithm first computes (k_{ij}, d_{ij}) for each arc. Once all (k_{ij}, d_{ij}) have been computed, the remaining steps are the same as that in Fu *et al*. (2010). For completeness, we provide it in the Appendix. Note that k_{ij} is computed using binary search. Hence, the complexity is $T_1 \times \log_2 D_{\max}$, where T_1 is the time required to compute $F(x)$ and $D_{\max}$ is

Table 7. Reservation and execution costs

	Options										
	1	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*Spot*
Reservation price c_i	10	9	8	7.1	6	5.1	4	3.1	2	1.1	0
Execution price h'_i	4.5	5.6	6.8	8.1	9.5	11	12.6	14.3	16.1	18.2	20

the maximum value of the demand. Once k_{ij} is found, d_{ij} can be computed easily by calculating the right-hand side integration with time T_2. Then we search for the shortest monotone path based on (k_{ij}, d_{ij}) and keep track of path distance for each node with all of its possible predecessors. We need to compare all of the k_{ij} along the path to ensure monotonicity. Given (k_{ij}, d_{ij}), the complexity of finding the shortest monotone distance to node n is $O(n^3)$. Please see the algorithm in the Appendix.

Given that the distribution function of demand is available, if we treat T_1 and T_2 as constants (for example, for a normal distribution or a uniform distributions, we can obtain $F(x)$ and d_{ij} in constant time for each i and j and hence the total time needed is $O(n^2)$), the complexity of the shortest monotone path algorithm is $O[n^2(\log_2 D_{\max} + n)]$, where n denotes the number of option suppliers.

Remark 5. The optimal solution to the PRMS would vary when the fixed costs of the options change. To illustrate this, we compare the minimum cost obtained using our algorithm and that by using the procurement strategy without considering the fixed ordering cost. In our sample, we choose 10 options (and the spot market), whose reserving costs and execution costs are listed in Table 7.

Assume that the demand follows a normal distribution $D \sim N(1000, 500^2)$. To simplify our analysis, we suppose that the fixed ordering costs of all options are identical. Figure 3 shows the comparison of the cost of the optimal solution and that of using the solution without considering fixed ordering costs, when the fixed ordering costs changes. As we can see from the figure, the error can be large when the fixed setup cost is high.

Remark 6. Fu *et al.* (2010) show that under the condition that neither a fixed ordering nor capacity constraint is imposed then if we only use two contracts instead of using the optimal solution, the errors are rather stable and the average errors on numerical experiments are less than 2%. On the other hand, they also show that the worst-case error bound of deleting a contract can be arbitrarily large. We believe that if the fixed ordering cost is small and capacity is large, then the results in Fu *et al*. (2010) can be applied to our case. On the other hand, if the capacity is small, then intuitively we know that deleting any contract can incur huge error.

Example 3: We consider the case with 10 suppliers offering option contracts and compare two cases (with and without fixed ordering costs) to see how the fixed costs affect the optimal procurement solution. The distributions of demand and spot price are the same as those in Example 2.

As can be observed from Table 8, some dominated option contracts are active in the optimal solution when the fixed costs are non-zero. Moreover, these active contracts can

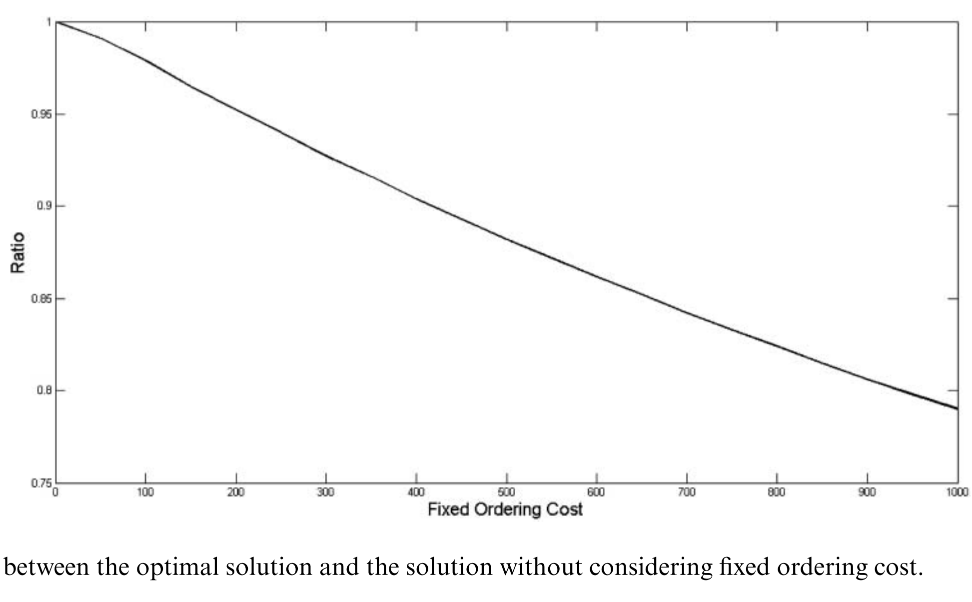

Fig. 3. Comparison between the optimal solution and the solution without considering fixed ordering cost.

Table 8. Option procurement with fixed ordering costs

Option	*h (execution cost)*	*c (reservation cost)*	*Fixed ordering cost I*	*Optimal reservation quantity*	*Fixed ordering cost II*	*Optimal reservation quantity*
1	4.5	10	0	332.4	80	332.4
2	5.6	9	0	183.9	80	314.4
3	6.8	8	0	160.8	50	0
4	8.1	7.1	0	0	0	105.6
5	9.5	6	0	136.1	30	0
6	11	5.1	0	0	10	163.6
7	12.6	4	0	79.6	60	0
8	14.3	3.1	0	0	10	0
9	16.1	2	0	104.1	70	0
10	18.2	1.1	0	0.4	30	0
Spot market	20	0				

form a new lower convex hull if the contracts lying below them are removed. This is consistent with the property described in Equations (7) and (8).

7. Conclusions and discussion

This article models the PRM-CK problem. Effective supply option contracts play an important role in reducing cost for a buyer, especially when demand is uncertain and the spot market price is highly volatile. When a number of suppliers in the market provide option contracts specifying different price terms, finding the optimal procurement strategy is both an interesting and challenging problem for the buyer.

We have considered three cases and develop frameworks for the design of an optimal supply contract portfolio in a single-buyer, single-period environment and show how the capacity constraints and fixed ordering costs affect the optimal procurement solution. Incorporating capacities and fixed ordering costs is important because (i) (all suppliers have limited capacities and a buyer incurs fixed ordering costs from maintaining relationships with suppliers, preparing legal documents, etc.; and (ii) the introduction of capacity constraints and fixed ordering costs significantly affects the optimal ordering policy. We mainly focus on the solution method and describe the characteristics of the procurement policy. In this article, we mainly focus on, one product. For the problem with multiple products, different products may have different demand patterns. Generally speaking, the option portfolio approach allows the buyer to meet the demand of each product with an appropriate policy. For example, the buyer would sign a lower-cost yet less flexible contract for a product with a known demand, and sign a higher-cost yet more flexible contract for the product with a volatile demand.

This article certainly has room for improvement. First, for the PRM-CK problem we assume that the ordering cost is buyer dependent and identical for all option contracts. More work is needed to find the optimal procurement approach if this assumption is relaxed. We also assume that the buyer is risk neutral and focus on minimizing the expected value of the cost function. One interesting issue is how to take the risk explicitly into consideration and how the risk attitude affects the portfolio procurement decision under the cases discussed.

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Appendix

Proof of Theorem 3. It is straightforward to see that the solution obtained from the Walk-and-Stop algorithm satisfies requirements 1 and 2 of the PRMC(S,I) problem. It remains to prove that it satisfies requirement 3 also; i.e., the optimality of the solution. For simplicity, we prove the following claim first.

Claim 1. If $C(q_1,\ldots,q_i,\ldots,q_j+1,\ldots,q_n) < C(q_1,\ldots,q_i+1,\ldots,q_j,\ldots,q_n)$ and $q'_k \geq q_k$ for every $i \leq k < j$, we have:

$$C(q_1,\ldots,q'_i,q'_{i+1},\ldots,q'_{j-1},q_j+1,\ldots,q_n) < C(q_1,\ldots,q'_i+1,q'_{i+1},\ldots,q'_{j-1},q_j,\ldots,q_n).$$

Proof of Claim 1. Let $A = \sum_{k=1}^{i-1} q_k$ and denote:

$$\begin{aligned}
&C(q_1,\ldots,q_i+1,q_{i+1},\ldots,q_n) \\
&\quad - C(q_1,\ldots,q_i,q_{i+1}+1,\ldots,q_n) \\
&\quad = c_i - c_{i+1} + (h'_i - h'_{i+1})\Pr(D > A+q_i) \\
&\quad = B_i \\
&\ldots\ldots\ldots \\
&C(q_1,\ldots,q_{j-1}+1,q_j,\ldots,q_n) \\
&\quad - C(q_1,\ldots,q_{j-1},q_j+1,\ldots,q_n) \\
&\quad = c_{j-1} - c_j + (h'_{j-1} - h'_j) \\
&\qquad Pr(D > A+q_i+q_{i+1}+\cdots+q_{j-1}) \\
&\quad = B_{j-1}.
\end{aligned}$$

It follows that:

$$\sum_{k=i}^{j-1} B_k = C(q_1,\ldots,q_i+1,\ldots,q_j,\ldots,q_n) - C(q_1,\ldots,q_i,\ldots,q_j+1,\ldots,q_n) > 0.$$

Next, we set

$$\begin{aligned}
&C(q_1,\ldots,q'_i+1,q'_{i+1},\ldots,q'_{j-1},q_j,q_{j+1},\ldots,q_n) \\
&\quad - C(q_1,\ldots,q'_i,q'_{i+1}+1,\ldots,q'_{j-1},q_j,\ldots,q_n) \\
&\quad = c_i - c_{i+1} + (h'_i - h'_{i+1})\Pr(D > A+q'_i) \\
&\quad = C_i \\
&\ldots\ldots\ldots \\
&C(q_1,\ldots,q_i,q'_{i+1},\ldots,q'_{j-1}+1,q_j,\ldots,q_n) \\
&\quad - C(q_1,\ldots,q_i,q'_{i+1},\ldots,q'_{j-1},q_j+1,\ldots,q_n) \\
&\quad = c_{j-1} - c_j + (h'_{j-1} - h'_j) \\
&\qquad \Pr(D > A+q'_i+q'_{i+1}+\cdots+q'_{j-1}) \\
&\quad = C_{j-1}.
\end{aligned}$$

For every $i \leq k \leq j-1$, we have

$$\begin{aligned}
C_k - B_k &= (h'_{k+1} - h'_k)\Pr(A+q_i+q_{i+1}+\cdots+q_k \\
&\qquad < D \leq A+q'_i+q'_{i+1}+\cdots+q'_k) \\
&\geq 0.
\end{aligned}$$

It follows that:

$$\sum_{k=i}^{j-1} C_k \geq \sum_{k=i}^{j-1} B_k > 0.$$

Since

$$\sum_{k=i}^{j-1} C_k = C(q_1, \ldots, q'_i + 1, q'_{i+1}, \ldots, q'_{j-1}, q_j, \ldots, q_n) - C(q_1, \ldots, q'_i, q'_{i+1}, \ldots, q'_{j-1}, q_j + 1, \ldots, q_n) > 0,$$

we are done. ■

Now we come back to the proof of Theorem 3. To prove the optimality of the solution, it suffices to show that any other solution is not optimal, and since there exists an optimal solution (the number of solutions is finite), we can prove the theorem. Formally, suppose that there is a reservation strategy $q^* = (q_1^*, q_2^*, \ldots, q_{n+1}^*)$ such that for every $i \in S$, $q_i^* = Q_i$ and $\sum_{i=1, i\notin S}^{n+1} q_i^* = I$. Suppose that $q^* \neq q$. We shall prove that q^* is not an optimal solution to the PRMC(S,I) problem, where q is the output from Walk-and-Stop algorithm.

Since $q^* \neq q$, let k be the smallest index such that $q_k^* \neq q_k$. It could be seen that $k \notin S$. Now we consider the following two possibilities:

1. $q_k^* > q_k$.

Let j be the smallest number such that $j > k$, $j \notin S$ and $q_j > 0$. Note that such j must exist since $\sum_{i=1, i\notin S}^{n+1} q_i^* = \sum_{i=1, i\notin S}^{n+1} q_i$.

Claim 2. $C(q_1^*, \ldots, q_k^* - 1, \ldots, q_j^* + 1, \ldots, q_{n+1}^*) < C(q_1^*, \ldots, q_{n+1}^*)$.

Proof of Claim 2. Based on the induction hypothesis and the property of the Walk-and-Stop algorithm, we could see that:

$$C(q_1, \ldots, q_k, \ldots, q_{j-1}, 1, v_{j+1}, \ldots, v_n) < C(q_1, \ldots, q_k + 1, \ldots, q_{j-1}, 0, v_{j+1}, \ldots, v_n),$$

where $v_d = Q_d$ when $d \in S$, and zero otherwise.
That is, when $(q_1, \ldots, q_k)$ is already reserved, it would cost less to reserve the next unit from j than from k. Based on Claim 1, we have

$$C(q_1, \ldots, q_k, q_{k+1}^*, \ldots, q_{j-1}^*, 1, v_{j+1}, \ldots, v_n) < C(q_1, \ldots, q_k + 1, q_{k+1}^*, \ldots, q_{j-1}^*, 0, v_{j+1}, \ldots, v_n).$$

This is because $q_i < q_i^*$ when $k < i < j$; $q_i = q_i^*$ when $i \in S$; and $q_i = 0$ otherwise.
Next, based on Claim 1 again, we obtain:

$$C(q_1, \ldots, q_k^* - 1, q_{k+1}^*, \ldots, q_{j-1}^*, 1, v_{j+1}, \ldots, v_n) < C(q_1, \ldots, q_k^*, q_{k+1}^*, \ldots, q_{j-1}^*, 0, v_{j+1}, \ldots, v_n),$$

and it follows that:

$$C(q_1, \ldots, q_k^* - 1, q_{k+1}^*, \ldots, q_{j-1}^*, 1, 0, \ldots, 0) < C(q_1, \ldots, q_k^*, q_{k+1}^*, \ldots, q_{j-1}^*, 0, \ldots, 0).$$

This is because in both strategies $(q_1, \ldots, q_k^* - 1, q_{k+1}^*, \ldots, q_{j-1}^*, 1, v_{j+1}, \ldots, v_n)$ and $(q_1, \ldots, q_k^*, q_{k+1}^*, \ldots, q_{j-1}^*, 0, v_{j+1}, \ldots, v_n)$, the total reservation quantities of the first j options are the same; therefore, the costs associated with the last $n - j$ options are the same: Let $M = q_1 + \cdots + q_k^* + \cdots + q_{j-1}^* + 1$, and the cost associated with the last $n - j$ options would be

$$c_{j+1}v_{j+1} + h_{j+1}\sum_{t=0}^{v_{j+1}} \Pr(D \geq M + t) + c_{j+2}v_{j+2} + h_{j+2}\sum_{t=0}^{v_{j+2}} \Pr(D \geq M + v_{j+1} + t) + \cdots.$$

Subtracting this part from both sides and adding the new cost for spot purchase, the result follows. For the same reason:

$$C(q_1, \ldots, q_k^* - 1, \ldots, q_j^* + 1, \ldots, q_n^*) < C(q_1, \ldots, q_k^*, \ldots, q_j^*, \ldots, q_n^*).$$

Note that $q_i = q_i^*$ when $i < k$, and in this way we have proved Claim 2, which implies that q^* is not an optimal solution. (Note that in the PRMC(S,I) problem, there is no capacity constraint on suppliers not in S.)

1. $q_k^* < q_k$.

The proof is similar to the proof of the case $q_k^* > q_k$. Here, let j be the smallest number such that $j > k$, $j \notin S$, and $q_j^* > 0$. The existence of such j is clear. Following a similar argument, we could prove that:

$$C(q_1^*, \ldots, q_k^* + 1, \ldots, q_j^* - 1, \ldots, q_{n+1}^*) < C(q_1^*, \ldots, q_{n+1}^*).$$

In this way, we could show that q^* is not the optimal solution. It follows that q is the optimal solution, and our proof is complete. ■

Proof of Theorem 4. Suppose that we run the Walk-and-Stop algorithm and *stop* it when we find the first contract, say k, that is not in S and saturated (i.e., $q_k \geq Q_k$). Denote the current output by $(q_1, \ldots, q_k, v_{k+1}, \ldots, v_n)$. Note that every contract $l > k$ is either not reserved ($v_l = 0$) or is in S ($v_l = Q_l$).

Claim 3. Let $q^* = (q_1^*, \ldots, q_n^*)$ be the optimal solution to the PRMC problem. Then $q_i^* \geq q_i$ for every $1 \leq i \leq k$.

Proof* of *Claim 3. Suppose that Claim 3 holds for 1 to $i - 1$. We shall prove that it also holds for i. The proof is trivial when $q_i = 0$ or $i \in S$, so we simply assume that $q_i > 0$ and $i \notin S$. Suppose that $q^* = (q_1^*, \ldots, q_n^*)$ is the optimal solution, with $q_i^* < q_i$. We shall prove that q^* is not optimal, which contradicts the assumption. We take the following two cases:

Case 1: There is no j such that $j < i, j \notin S$, and $q_j > 0$.

Here we pick up the smallest l such that $l \neq i$, $l \notin S$, and $q_l^* > 0$. We shall prove that q^* can be improved by reserving one unit less from q_l^* and one more unit from q_i^*.
Case 1.1: $l > i$.

By the optimality of the Walk-and-Stop algorithm and the assumption that $q_i^* < q_i$, we have

$$C(v_1, \ldots, q_i^* + 1, \ldots, 0(contractl), \ldots, v_n) < C(v_1, \ldots, q_i^*, \ldots, 1(contractl), \ldots, v_n),$$

where $v_i = Q_i$ when $i \in S$ and zero otherwise. Note that $v_l = 0$ because $l \notin S$, as we assumed.
Since for any $d < l$ and $d \neq i$, there is $q_d^* = v_d$, we have

$$C(q_1^*, \ldots, q_i^* + 1, \ldots, q_{l-1}^*, 0, \ldots, v_n) < C(q_1^*, \ldots, q_i^*, \ldots, q_{l-1}^*, 1, \ldots, v_n).$$

Since for both strategies the total reservation quantities for the first l options are the same, and the reservation quantities of the last $n - l$ options are same, we replace the reservation of last $n - l$ options with zero in both sides, the inequality still holds:

$$C(q_1^*, \ldots, q_i^* + 1, \ldots, q_{l-1}^*, 0, \ldots, 0) < C(q_1^*, \ldots, q_i^*, \ldots, q_{l-1}^*, 1, \ldots, 0),$$

and

$$C(q_1^*, \ldots, q_i^* + 1, \ldots, q_{l-1}^*, q_l^* - 1, \ldots, 0) < C(q_1^*, \ldots, q_i^*, \ldots, q_{l-1}^*, q_l^*, \ldots, 0),$$
$$C(q_1^*, \ldots, q_i^* + 1, \ldots, q_{l-1}^*, q_l^* - 1, \ldots, q_n^*) < C(q_1^*, \ldots, q_i^*, \ldots, q_{l-1}^*, q_l^*, \ldots, q_n^*).$$

This contradicts the optimality of q^* or the assumption that $q_i^* < q_i \leq Q_i$.
Case 1.2: $l < i$**.** By the optimality of the Walk-and-Stop algorithm and the assumption that $q_i^* < q_i$, we have

$$C(v_1, \ldots, 0(contractl), \ldots, q_i^* + 1, \ldots, v_n) < C(v_1, \ldots, 1(contractl), \ldots, q_i^*, \ldots, v_n).$$

Following Claim 1 and the induction hypothesis, we could prove that:

$$C(q_1^*, \ldots, q_l^* - 1, \ldots, q_i^* + 1, \ldots, v_n) < C(q_1^*, \ldots, q_l^*, \ldots, q_i^*, \ldots, v_n).$$

Hence,

$$C(q_1^*, \ldots, q_l^* - 1, \ldots, q_i^* + 1, \ldots, q_n^*) < C(q_1^*, \ldots, q_l^*, \ldots, q_i^*, \ldots, q_n^*),$$

which contradicts the optimality of q^* or the assumption that $q_i^* < q_i \leq Q_i$.
Case 2: Let j be the largest number such that $j < i, j \notin S$, and $q_j > 0$.
Based on the optimality of the Walk-and-Stop algorithm, we have

$$C(q_1, \ldots, q_j, \ldots, q_i, v_{i+1}, \ldots, v_n) < C(q_1, \ldots, q_j + 1, \ldots, q_i - 1, v_{i+1}, \ldots, v_n)$$

(if $q_j = Q_j$, the algorithm would stop at $j < i$, so $q_j < Q_j$).
It follows that:

$$C(q_1, \ldots, q_j, \ldots, q_i, 0, \ldots, 0) < C(q_1, \ldots, q_j + 1, \ldots, q_i - 1, 0, \ldots, 0).$$

And note that since $q_1 + \cdots + q_j + \cdots + q_i = q_1 + \cdots + (q_j + 1) + \cdots + (q_i - 1)$:

$$C(q_1, \ldots, q_j, \ldots, q_i + 1, 0, \ldots, 0) < C(q_1, \ldots, q_j + 1, \ldots, q_i, 0, \ldots, 0)$$

and

$$C(q_1, \ldots, q_j, \ldots, q_i + 1, \ldots, q_n) < C(q_1, \ldots, q_j + 1, \ldots, q_i, \ldots, q_n).$$

Case 2.1: $\sum_{d=1}^{j} q_d^* > \sum_{d=1}^{j} q_d$.
Now we shall prove that:

$$C(q_1^*, \ldots, q_j^* - 1, \ldots, q_i + 1, \ldots, q_n) < C(q_1^*, \ldots, q_j^*, \ldots, q_i, \ldots, q_n).$$

Let $A = \sum_{d=1}^{j} q_d$, $B = \sum_{d=1}^{j} q_d^* - 1$:

$$A_j = (c_j - c_{j+1}) + (h'_j - h'_{j+1})\Pr(D > A),$$
$$B_j = (c_j - c_{j+1}) + (h'_j - h'_{j+1})\Pr(D > B).$$

Since $B \geq A$, $h'_j < h'_{j+1}$, we have $B_j \geq A_j$.
Define $A_{j+1}, \ldots, A_{i-1}$ and $B_{j+1}, \ldots, B_{i-1}$ similarly.
Next it follows that:

$$\sum_{d=j}^{i-1} A_d = C(q_1, \ldots, q_j + 1, \ldots, q_i, \ldots, q_n) - C(q_1, \ldots, q_j, \ldots, q_i + 1, \ldots, q_n) > 0,$$
$$\sum_{d=j}^{i-1} B_d \geq \sum_{d=j}^{i-1} A_d > 0.$$

Thus,

$$C(q_1^*, \ldots, q_j^* - 1, \ldots, q_i + 1, \ldots, q_n) < C(q_1^*, \ldots, q_l^*, \ldots, q_i, \ldots, q_n).$$

Further based on Claim 1, we could prove that:

$$C(q_1^*, \ldots, q_j^* - 1, \ldots, q_i^* + 1, \ldots, q_n^*) < C(q_1^*, \ldots, q_j^*, \ldots, q_i^*, \ldots, q_n^*).$$

This contradicts the optimality of q^* or the assumption that $q_i^* < q_i \leq Q_i$.
Case 2.2: $\sum_{d=1}^{j} q_d^* \leq \sum_{d=1}^{j} q_d$.
Note that based on the induction hypothesis, we have $\sum_{d=1}^{j} q_d^* \geq \sum_{d=1}^{j} q_d$; thus, we have

$$\sum_{d=1}^{j} q_d^* = \sum_{d=1}^{j} q_d.$$

Based on the induction hypothesis, $q_l^* = q_l$ when $l < j$.
Remember that j is the largest number such that $j < i$, $j \notin S$, and $q_j > 0$. In this way, we have

$$\sum_{d=j+1}^{i-1} q_d^* \geq \sum_{d=j+1}^{i-1} q_d.$$

We claim that:

$$\sum_{d=j+1}^{i-1} q_d^* = \sum_{d=j+1}^{i-1} q_d.$$

Otherwise, let $j < l$ be the smallest number such that $q_l^* > q_l$. Based on the assumption ($\sum_{d=j+1}^{i-1} q_d^* > \sum_{d=j+1}^{i-1} q_d$), we have $j < l < i$. Then, by the optimality of the Walk-and-Stop algorithm, we have

$$\begin{aligned} &C(q_1, \ldots, q_l, \ldots, q_i^* + 1, v_{i+1}, \ldots, v_n) \\ &\quad < C(q_1, \ldots, q_l + 1, \ldots, q_i^*, v_{i+1}, \ldots, v_n). \end{aligned}$$

Further, we could prove that:

$$\begin{aligned} &C(q_1^*, \ldots, q_l^* - 1, \ldots, q_i^* + 1, \ldots, q_n^*) \\ &\quad < C(q_1^*, \ldots, q_l^*, \ldots, q_i^*, \ldots, q_n^*), \end{aligned}$$

which gives a contradiction. In this way, we have

$$\sum_{d=j+1}^{i-1} q_d^* = \sum_{d=j+1}^{i-1} q_d.$$

Combining it with:

$$\sum_{d=1}^{j} q_d^* = \sum_{d=1}^{j} q_d,$$

we come up with:

$$\sum_{d=1}^{i-1} q_d^* = \sum_{d=1}^{i-1} q_d.$$

Based on the induction hypothesis, we have $q_l^* = q_l$ when $l < i$.

Let $l > i$ be the smallest number such that $q_l^* > 0$ and $l \notin S$ based on the optimality of the Walk-and-Stop algorithm:

$$\begin{aligned} &C(q_1^*, \ldots, q_i^* + 1, v_{i+1}, \ldots, 0\ (contract l), \ldots, v_n) \\ &\quad < C(q_1^*, \ldots, q_i^*, v_{i+1}, \ldots, 1, \ldots, v_n). \end{aligned}$$

Note that for $i < d < l$, we have $v_d = q_d^*$; thus,

$$\begin{aligned} &C(q_1^*, \ldots, q_i^* + 1, q_{i+1}^*, \ldots, 0\ (contract l), \ldots, v_n) \\ &\quad < C(q_1^*, \ldots, q_i^*, q_{i+1}^*, \ldots, 1, \ldots, v_n), \end{aligned}$$

and finally

$$\begin{aligned} &C(q_1^*, \ldots, q_i^* + 1, \ldots, q_l^* - 1, \ldots, q_n^*) \\ &\quad < C(q_1^*, \ldots, q_i^*, \ldots, q_l^*, \ldots, q_n^*), \end{aligned}$$

contradicting the optimality of q^*.

Theorem 4 follows immediately from Claim 3. ■

Proof of Theorem 5. We prove the theorem in two parts to show that the optimal solution must guarantee the rationality and correctness of Equations (7) and (8).

1. Let

$$\alpha_{i_k} = \frac{(c_{i_k} - c_{i_{k+1}})}{\left(h'_{i_{k+1}} - h'_{i_k}\right)},$$

where i_k and i_{k+1} are two consecutive active contracts. Conditions (7) and (8) require that $\alpha_{i_k} \in [0, 1]$ and $\{\alpha_{i_k}\}$ are a non-increasing series. We call it the *convex hull property* and claim that the optimal solution of PRMS must have this property. Any solution that does not have the convex hull property cannot be the optimal solution. A solution A has active contract set $\{i_1 < i_2 < \cdots < i_{|A|}\}$. $q_{i_k}^*$ denotes the reservation quantity of option contract i_k, $k = 1, \ldots, |A|$. We consider three conditions.

1. If $\alpha_{i_k} < 0$ for certain k, $c_{i_k} < c_{i_{k+1}}$. We modify solution A to B by setting:

   $$\tilde{q}_{i_{k+1}} = 0, \tilde{q}_{i_k} = q_{i_k}^* + q_{i_{k+1}}^*.$$

   For any realization of demand and spot price, modify the second-stage decision to

   $$\begin{aligned} &\tilde{x}_{i_{k+1}}(D, P_s) = 0, \ \textit{and}\ \tilde{x}_{i_k}(D, P_s) \\ &\quad = x_{i_k}(D, P_s) + x_{i_{k+1}}(D, P_s). \end{aligned}$$

   The modified solution B has a smaller procurement cost than A, as the setup cost of option contract i_{k+1} is removed, $c_{i_k} \times \tilde{q}_{i_k} < c_{i_k} \times q_{i_k}^* + c_{i_{k+1}} \times \cdot q_{i_{k+1}}^*$, and $h'_{i_k} \times \tilde{x}_{i_k}(D, P_s) < h'_{i_k} \times x_{i_k}(D, P_s) + \times \cdot x_{i_{k+1}}(D, P_s)$. Thus, solution A cannot be the optimal solution.

2. If $\alpha_{i_k} > 1$ for certain k, $c_{i_k} - c_{i_{k+1}} > h'_{i_{k+1}} - h'_{i_k} > 0$, and $c_{i_k} + h'_{i_k} > c_{i_{k+1}} + h'_{i_{k+1}}$. We modify the solution A to B by

   $$\tilde{q}_{i_k} = 0, \tilde{q}_{i_{k+1}} = q_{i_{k+1}}^* + q_{i_k}^*$$

   and modify the second stage decision as

   $$\begin{aligned} &\tilde{x}_{i_k}(D, P_s) = 0, \tilde{x}_{i_{k+1}}(D, P_s) \\ &\qquad = x_{i_{k+1}}(D, P_s) + x_{i_k}(D, P_s). \end{aligned}$$

   The modified solution B has a smaller procurement cost than A, as the setup cost for option contract i_k is removed, and

   $$\begin{aligned} &c_{i_k} \times q_{i_k}^* + c_{i_{k+1}} \times q_{i_{k+1}}^* + h'_{i_k} \times x_{i_k} + h'_{i_{k+1}} \times x_{i_{k+1}} \\ &\quad = c_{i_k} \times (q_{i_k}^* - x_{i_k}) + (c_{i_k} + h'_{i_k}) \times x_{i_k} + c_{i_{k+1}} \\ &\qquad \times (q_{i_{k+1}}^* - x_{i_{k+1}}) + (c_{i_{k+1}} + h'_{i_{k+1}}) \times x_{i_{k+1}} \\ &\quad > c_{i_{k+1}} \times (q_{i_k}^* - x_{i_k}) + (c_{i_{k+1}} + h'_{i_{k+1}}) \times x_{i_k} + c_{i_{k+1}} \\ &\qquad \times (q_{i_{k+1}}^* - x_{i_{k+1}}) + (c_{i_{k+1}} + h'_{i_{k+1}}) \times x_{i_{k+1}} \\ &\quad = c_{i_{k+1}} \times \tilde{q}_{i_{k+1}} + h'_{i_{k+1}} \times \tilde{x}_{i_{k+1}}(D, P_s). \end{aligned}$$

3. If $\alpha_{i_k} \in [0, 1]$ but $\{\alpha_{i_k}\}$ are not non-increasing series, e.g., $0 \le \alpha_{i_k} < \alpha_{i_{k+1}} \le 1$, it is easy to see that option i_{k+1} is dominated by options i_k and i_{k+2}. Similar to the proof above, we can modify solution A to B by reducing the reservation quantity of contract i_{k+1} to be zero and adding certain reservation quantity to contract i_k and i_{k+2}. We also modify the second-stage decision appropriately. B is a better solution than A, since the setup cost of contact i_{k+1} is gone and the reservation as well as execution cost is also reduced.

2. Every solution of PRMS is denoted by the set of active contracts and the corresponding reserving quantity. For any solution, say A, that does not satisfy the equality in Equations (7) or (8), we prove its non-optimality by perturbing it to a better one, say B, without changing the set of active contracts.

For solution A, if Condition (7) is not satisfied, for example "$<$" instead of "$=$" holds for certain k, then we change solution A to B by increasing $q^*_{i_k}$ by $\epsilon(\epsilon > 0)$ and decreasing $q^*_{i_{k+1}}$ by ϵ, while keeping $q^*_{i_{k+1}} - \epsilon > 0$. The total procurement cost of B will change by the following amount.

1. Total setup cost does not change, since the active contracts do not change.
2. Reservation cost will change by $\epsilon(c_{i_k} - c_{i_{k+1}})$.
3. In the event that $D \le \sum_{r=1}^{k} q^*_{i_r}$ or $P_s \le h_{i_k}$, the change does not affect the execution cost, and thus has no effect on the total procurement cost.
4. In the event that $D > \sum_{r=1}^{k} q^*_{i_r}$ and $P_s > h_{i_k}$, the change in execution cost will be $\epsilon(h_{i_k} - P_s)$ if $h_{i_k} < P_s < h_{i_{k+1}}$ and $\epsilon(h_{i_k} - h_{i_{k+1}})$ if $h_{i_{k+1}} < P_s$. The change in total procurement cost will be

$$\epsilon\left\{(c_{i_k} - c_{i_{k+1}}) - \int_{\sum_{r=1}^{k} q^*_{i_r}}^{\infty}\left[\int_{h_{i_k}}^{h_{i_{k+1}}}(P_s - h_{i_k}) + \int_{h_{i_{k+1}}}^{\infty}(h_{i_{k+1}} - h_{i_k})\right] f(D, P_s)\mathrm{d}P_s\mathrm{d}D\right\}$$
$$= \epsilon\left\{(c_{i_k} - c_{i_{k+1}}) - P\left(D > \sum_{r=1}^{k} q^*_{i_r}\right)\left(h'_{i_{k+1}} - h'_{i_k}\right)\right\} < 0.$$

On the other hand, if "$>$" instead of "$=$" holds in Condition (7), we change solution A to B by decreasing $q^*_{i_k}$ by $\epsilon(\epsilon > 0)$ and increasing $q^*_{i_{k+1}}$ by ϵ, while keeping $q^*_{i_k} - \epsilon > 0$. Solution B still has a smaller procurement cost than solution A.

Similarly, if Condition (8) is not satisfied, say "$<$" instead of "$=$" holds, we can change solution A to B by increasing $q^*_{i_{|A|}}$ by ϵ and decreasing z^* by ϵ while keeping $z^* - \epsilon > 0$. The change in total procurement cost will be

$$\epsilon\left\{c_{i_{|S|}} - P\left(D > \sum_{r=1}^{|S|} q^*_{i_r}\right)\left(E(P_s) - h'_{i_{|S|}}\right)\right\} < 0.$$

On the other hand, if Condition (8) is not satisfied, say "$>$" instead of "$=$" holds, we can change solution A to B by decreasing $q^*_{i_{|A|}}$ by ϵ and increasing z^* by ϵ while keeping $q^*_{i_{|A|}} - \epsilon > 0$. Solution B still has a smaller procurement cost than solution A.

In summary, the optimal solution of PRMS must satisfy Conditions (7) and (8). ■

Proof of Theorem 6. Given a reservation solution, the optimal execution strategy in stage 2 is just a greedy fashion, and the total cost can be deduced easily.

$$\begin{aligned}
C_S(q^*) &= c_1q_1^* + c_2q_2^* + \cdots + c_Nq_N^* + K_1 + K_2 + \cdots + K_N \\
&\quad + E(D)\int_0^{h_1} P_s g(P_s)\mathrm{d}P_s \\
&\quad + \int_{h_1}^{h_2}\{h_1E[\min(D, q_1^*)] + P_sE[D - \min(D, q_1^*)]\}g \\
&\quad \times (P_s)\mathrm{d}P_s \\
&\quad \cdot \\
&\quad \cdot \\
&\quad + \int_{h_{N-1}}^{h_N}\Bigg\{h_1E[\min(D, q_1^*)] \\
&\quad + \sum_{i=2}^{N-1} h_iE\left[\min\left(D, \sum_{r=1}^{i} q_r^*\right) - \min\left(D, \sum_{r=1}^{i-1} q_r^*\right)\right] \\
&\quad + P_sE\left[D - \min\left(D, \sum_{r=1}^{N-1} q_r^*\right)\right]\Bigg\} g(P_s)\,\mathrm{d}P_s \\
&\quad + \int_{h_N}^{\infty}\Bigg\{h_1E[\min(D, q_1^*)] \\
&\quad + \sum_{i=2}^{N} h_iE\left[\min\left(D, \sum_{r=1}^{i} q_r^*\right) - \min\left(D, \sum_{r=1}^{i-1} q_r^*\right)\right] \\
&\quad + P_sE\left[D - \min\left(D, \sum_{r=1}^{N} q_r^*\right)\right]\Bigg\} g(P_s)\,\mathrm{d}P_s.
\end{aligned}$$

Note that

$$\begin{aligned}
c_1q_1^* + c_2q_2^* + \cdots + c_Nq_N^* &= (c_1 - c_2)\,q_1^* + (c_2 - c_3)\,(q_1^* + q_2^*) \\
&\quad + \cdots + (c_{N-1} - c_N)\sum_{i=1}^{N-1} q_i^*
\end{aligned}$$

$$h'_i \equiv E\,[\min(h_i, P_s)] = \int_0^{h_i} P_s g(P_s)\,\mathrm{d}P_s + \int_{h_i}^{\infty} h_i g(P_s)\,\mathrm{dP_s}$$

and

$$E\left[\min\left(D, \sum_{r=1}^{i} q_r^*\right)\right] = \int_0^{\sum_{r=1}^{i} q_r^*} Df(D)\mathrm{d}D + \Pr\left(D > \sum_{r=1}^{i} q_r^*\right) \times \left(\sum_{r=1}^{i} q_r^*\right).$$

Rearranging the terms and using Properties (7) and (8), we obtain:

$$\begin{aligned}
C_S(q^*) &= K_1 + K_2 + \cdots + K_N + E(D)\int_0^{h_1} P_s g(P_s)\,\mathrm{d}P_s \\
&\quad + \sum_{i=1}^{N-1}\Bigg[h_i \int_{h_i}^{\infty} g(P_s)\,\mathrm{d}P_s \times \int_0^{\sum_{r=1}^{i} q_r^*} Df(D)\mathrm{d}D \\
&\quad - h_{i+1}\int_{h_{i+1}}^{\infty} g(P_s)\,\mathrm{d}P_s \times \int_0^{\sum_{r=1}^{i} q_r^*} Df(D)\mathrm{d}D \\
&\quad + \int_{h_i}^{h_{i+1}} P_s g(P_s)\,\mathrm{d}P_s \times \int_{\sum_{r=1}^{i} q_r^*}^{\infty} Df(D)\mathrm{d}D\Bigg] \\
&= K_1 + K_2 + \cdots + K_N + E(D)\int_0^{h_1} P_s g(P_s)\,\mathrm{d}P_s \\
&\quad + \sum_{i=1}^{N-1}\Bigg[h_i' \int_0^{\sum_{r=1}^{i} q_r^*} Df(D)dD \\
&\quad - h_{i+1}'\int_0^{\sum_{r=1}^{i} q_r^*} Df(D)dD \\
&\quad + \left(\int_{h_i}^{h_{i+1}} P_s g(P_s)\,\mathrm{d}P_s\right) E(D)\Bigg] \\
&= K_1 + K_2 + \cdots + K_N + E(D)E(P_s) \\
&\quad + \sum_{i=1}^{N-1}\Bigg[h_i' \int_0^{\sum_{r=1}^{i} q_r^*} Df(D)\mathrm{d}D \\
&\quad - h_{i+1}'\int_0^{\sum_{r=1}^{i} q_r^*} Df(D)\mathrm{d}D\Bigg] \\
&\quad + h_N'\int_0^{\sum_{r=1}^{N} q_r^*} Df(D)\mathrm{d}D - E(P_s)\int_0^{\sum_{r=1}^{N} q_r^*} Df(D)\mathrm{d}D \\
&= K_1 + \sum_{i=1}^{N-1}\Bigg(K_{i+1} + h_i' \int_0^{\sum_{r=1}^{i} q_r^*} Df(D)\mathrm{d}D \\
&\quad - h_{i+1}'\int_0^{\sum_{r=1}^{i} q_r^*} Df(D)\mathrm{d}D\Bigg) \\
&\quad + h_N'\int_0^{\sum_{r=1}^{N} q_r^*} Df(D)\mathrm{d}D + E(P_s)\int_{\sum_{r=1}^{N} q_r^*}^{\infty} Df(D)\mathrm{d}D
\end{aligned}$$

■

Algorithm 2 (The Shortest Monotone Path Algorithm).
Input: c_i, h_i, *and* K_i, $i = 1, \ldots, n$, $F(D)$, *and* $G(P_s)$.

Output: Optimal ordering quantities of each option contract; optimal procurement cost

Step 1: Construct the directed network.
Construct a graph with $n+2$ nodes. Nodes O and E are indexed as 0 and $n+1$, respectively. For each arc(i, j), $0 \le i < j \le n+1$, compute $k_{i,j}$ and $d_{i,j}$. For each node i, define array $dist(i)$ with length i, and let the tth element $dist(i, t)$ denote the shortest monotone distance from node O to node i with predecessor $t-1$.

Step 2: Find distance labels for each node with all possible predecessors.

```
Do
for i = 1, 2, ..., n + 1
  set dist(i, 1) = d_0i
end for;
for i = 2, 3, ..., n + 1
  for t = 2, 3, ..., i
    if t = 2
      set dist(i, t) = d_01 + d_1i
    else
      set min = dist(t − 1, 1)
        for tt = 2, 3, ..., t − 1
          if k_{tt−1,t−1} < k_{t−1,i} and dist(t − 1, tt) < min
            set min = dist(t − 1, tt)
          end if
        end for
        set dist(i, t) = min + d_{t−1,i}
    end if
  end for
end for;
```

Step 3: Trace back for the optimal solution.
Construct an empty array *PATH* with element denoting the node index. Let $n+1$ be the first element of *PATH*. Among all of the valid distance labels of node $n+1$, choose the one with the smallest label, say j. Then $dist(n+1, j)$ denotes the shortest-monotone distance that is equal to the optimal procurement cost. Let $pred = j - 1$ and $distpred = dist(n+1, j) - d_{j-1,n+1}$. Let *pred* be the first element of *PATH*.

```
while the first element in PATH >1
    for i = 1, ..., pred
      if dist(pred,i) = distpred
        set distpred = dist(pred,i) − d_{i−1, pred} and pred = i − 1
      end if
    end for
    put pred first in PATH
end while
```

The elements in *PATH* denote the index of the active option contracts in the optimal solution. The corresponding ordering quantity can be found from $k_{i,j}$.

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